# Features : No Black Box Sample

A K Kapoor

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**Abstract**

One of the important requirements is that a document contain no black boxes. The mathematical details are all made available right in the document. This does away with the need for having pen pencil and paper while learning something. You may sit back and read in a relaxed fashion.

However there is danger that one might loose sight of logical connections between different parts of an article. For easy identification and to demarcate the algebraic and routine computational details in a transparent fashion, it proposed to try out a format in which the extra details are supplied in an **"Anti-Black-BoX"** through scanned hand written notes merged as images.

The reader has a choice of going through the details, or leaving them out completely without a loss of continuity.

This sample also tests stating known results from other places with a gray background. This facilitates reading of the article without referring to any other reference. The **"Recall-BoX"** tells the you the results in a precise form, which you may, generally, remember but may not be able to state very precisely. Repeated encounters with such "'Recall Boxed Statements" and should help you in

- identifying and remembering the results;
- being aware of interconnections between different results.

## §1 Introduction

A derivation of Planck's law of black body radiation will be presented. I suppose you have come across Planck's law of black body radiation. It is a well known graph that flashes in my mind when ever any one mentions black body radiation or Planck's law.

There are several questions that you need to settle before we get onto derivation. The $X$- axis of the plot is of course the frequency of radiation.

But what exactly does $y-$ coordinate represent ? Is it intensity ? intensity of what? Is it energy ? energy of what? Or is it something else? How does one measure this

What is a black body? Is it a cavity enclosing radiation? Does the answer depend on the geometry of the cavity? Does it depend on the material of the cavity walls? If not what tells you that it does not?

OK it is energy density for a given frequency of photons. The answer is independent of cavity properties can be proved using second law of thermodynamics.

Tell me what measurement have to be carried out to get information about energy density ($\equiv$ energy /volume) in a small range, $\omega$ to $\omega + d\omega$, of frequency? Are you not keen to know the answers of these questions. Search your favourite books and see which book provides the answer.

A whole lot of similar questions can be asked about Maxwellian distribution of velocities in gases at temperature $T$. How does one verify this famous, successful result of kinetic theory? Are interested? Find out for yourself.

## §2 The Partition Function for Photons

Let us first recall what is going to be used in the derivation?

1. The radiation is treated as a gas (collection) of non-interacting particles.

2. The microstates of a photon gas are specified by a giving a numbers $\nu_1, \nu_2, \nu_3 \cdots$ for energy levels $\epsilon_1, \epsilon_2, \epsilon_3, \cdots$. One set of these values $\{\nu_r(j)\}$ specifies a micro state of the system. A sum over all micro states means summation over all possible values of $\nu_r$ which are all non-negative integers.

3. For every frequency, the photons have two polarizations. and energy of a micro state is given by
$$E = \sum_j \sum_r \nu_r(j)\epsilon_r \tag{1}$$
The sum over $j$ represents the sum over all and the sum over $r$ is sum over all single particle states (orbitals).

4. Boltzmann distribution the probability of a system having energy $E$ when it is kept at temperature $T$ is the Boltzmann factor $\Xi_j = \exp(-\beta\nu_r\epsilon_r) \equiv \mathcal{P}(\nu_\bigtriangledown)$. Here $\{\epsilon_r | r = 1, 2, 3, \cdots\}$ denote the single particle energies, where $\Xi_r$ is the partition function
$$\begin{aligned}\Xi_r &= \sum_{\nu_r(j)=0}^{\infty} \exp(-\beta\nu_r(j)\epsilon_r) &(2)\\ &= 1 + e^{-\beta\epsilon_r} + e^{-2\beta\epsilon_r} + e^{-3\beta\epsilon_r} + \cdots &(3)\\ &= \frac{1}{1-\exp(-\beta\epsilon_r)}. &(4)\end{aligned}$$
Here the sum is over all possible single particle state occupation numbers, 0,1,2,3, (for bosons) for state $r$.

5. The average number of particles $< n_r >$ in a single particle state with energy $\epsilon_r$ can be see to be given by
$$< n_r >= \sum_j \nu_r(j)\exp(-\beta\epsilon_r) \tag{5}$$
For a derivation of result see Details in ABBox1.

<u>Average occupation number for bosons:</u>

We note that

$$\nu \exp(-\beta\nu\epsilon_r) = -\frac{1}{\beta}\frac{\partial}{\partial\epsilon_r}\exp(-\beta\nu\epsilon_r)$$

So that

$$\langle n_r\rangle = \frac{1}{\Xi_r}\sum_{\nu_j=0}^{\infty}\nu_j \exp(-\beta\nu_j\epsilon_r)$$

$$= \frac{1}{\Xi_r}\left(\frac{-1}{\beta}\right)\frac{\partial}{\partial\beta}\sum_{\nu_j=1}^{\infty}\exp(-\beta\nu_j\epsilon_r)$$

$$= \frac{1}{\Xi_r}\left(\frac{-1}{\beta}\right)\frac{\partial}{\partial\beta}\Xi_r \quad \left(\text{Using definition of } \Xi_r\right)$$

$$= \cancel{\frac{1}{\Xi_r}}\ -\frac{1}{\beta}\frac{\partial}{\partial\beta}\ln\Xi_r \qquad \text{—— (1)}$$

Using expression $\Xi_r = \frac{1}{1-\exp(-\beta\epsilon_r)}$ gives

$$\left(\frac{-1}{\beta}\right)\frac{\partial}{\partial\beta}\ln\Xi_r = \frac{1}{\beta}\ln(1-\exp(-\beta\epsilon_r))$$

$$= \left(\frac{1}{\beta}\right)\frac{1}{1-\exp(-\beta\epsilon_r)}\times\exp(-\beta\epsilon_r)\times\beta\epsilon_r$$

$$= \frac{\exp(-\beta\epsilon_r)}{1-\exp(-\beta\epsilon_r)} = \frac{1}{\exp(\beta\epsilon_r)-1} \qquad \text{—(2)}$$

$\therefore$ Use (1) and (2)

$$\langle n_r\rangle = \frac{1}{\exp(\beta\epsilon_r)-1}$$

**Box.1** $\boxed{\text{Details of Eq.(5)}}$

# §3 Now Photons

Now let us ask ourselves what are the single particle energies $\epsilon_r$ for a gas of photons in cavity at temperature $T$? The A photon with frequency $\omega$ has energy $\hbar\omega$. Here the index $r$ is replaced by continuous variable $\omega$. We must now count the number of photon states in the frequency range $\omega$ and $\omega + d\omega$ inside a cavity of volume $V$. Notice for momentum range $p$ to $p + dp$ and volume $V$ the number of states is

$$2 \times \frac{V}{(2\pi\hbar)^3} = 2 \times \frac{V}{(2\pi\hbar)^3}4\pi p^2 dp \tag{6}$$

Here a factor of 2 has been included because photons have two states of polarization. Noting the energy momentum relation $p = E/c = \hbar\omega/c$, the above expression can be rearranged as

$$2 \times \frac{V}{(2\pi\hbar)^3}4\pi p^2 dp = \frac{(8\pi V)}{(2\pi\hbar)^3}\left(\frac{\hbar}{c}\right)^3\omega^2 d\omega = \frac{V\omega^2 d\omega}{\pi^2 c^3} \tag{7}$$

Therefore the number of photons in the frequency range $\omega$ and $\omega + d\omega$ is given by

$$N_\omega = \frac{V\omega^2 d\omega}{\pi^2 c^3}\frac{1}{\exp(\beta\hbar\omega) - 1}. \tag{8}$$

Hence the energy of the radiation in this frequency range is

$$E - \omega = N_\omega \hbar \omega. \tag{9}$$

If $u(\omega)$ stands for energy per unit volume per unit frequency range, then the above expression is equal to $u(\omega)d\omega V$, giving

$$\boxed{u(\omega) = \frac{\hbar \omega^3}{\pi^2 c^3 (\exp(\beta \hbar \omega) - 1)}.} \tag{10}$$