QM-09 Lecture Notes
Pictures in quantum mechanics
Heisenberg Picture*

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Abstract

The time evolution of states a quantum system is given by the time dependent Schrodinger equation. Besides this framework, called the Schrödinger picture, other scheme are possible.In the Heisenberg picture the observable evolve according to the equation

$$\frac{dX}{dt} = \frac{1}{i\hbar}[F, H]$$

This equation corresponds to the equation of motion in the Poisson bracket formalism.

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§1 Schrodinger picture

In most commonly used description of quantum mechanics, the time development is described by the time dependent Schrödinger equation.

$$i\hbar\frac{\partial|\psi t\rangle}{\partial t} = H|\psi t\rangle \tag{1}$$

where H is the Hamiltonian operator of the system. The dynamical variables are operators and do not evolve with time. This description of time evolution is known as the *Schrodinger picture* of quantum mechanics.

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Note that average values and probabilities are observable quantities, but not the wave function or the state vector. This fact allows to describe the time development in several possible ways. We will describe two alternate important ways of describing time development of a quantum system known as the Heisenberg picture and the Dirac picture.

We use subscript S to denote the Schrodinger picture states $|\psi\rangle_S$ and operators $X_S(q,p)$ or simply X_S.

To simplify present discussion, we will assume that the Hamiltonian is independent of time. The state vector at time t is given by

$$|\psi t\rangle_S = U(t,t_0)|\psi_{t_0}\rangle_S \tag{2}$$

where the time evolution operator is given by

$$U(t,t_0) = \exp(-iH(t-t_0)/\hbar). \tag{3}$$

Without loss of generality, we will set $t_0 = 0.$ and write

$$|\psi t\rangle_S = e^{-iHt/\hbar}|\psi_0\rangle \tag{4}$$

§2 Heisenberg Picture

The Heisenberg picture state vector is defined by

$$|\psi t\rangle_H = e^{iHt/\hbar}|\psi t\rangle_S = |\psi_0\rangle. \tag{5}$$

The Heisenberg state vector is independent of time and coincid with the state vector in the Schrödinger picture at initial time. The time development of the Heisenberg picture operators is defined so that the average value of any dynamical variable at time t in the Schródinger and Heisenberg pictures coincide. Thus we demand

$$_H\langle\psi t|X_H(t)|\psi t\rangle_H = \ _S\langle\psi t|X_S|\psi t\rangle_S \tag{6}$$

Substituting (4) and (5) gives

$$\langle\psi_0|X_H(t)|\psi_0\rangle = \langle\psi_0|e^{iHt/\hbar}\, X_S\, e^{-iHt/\hbar}|\psi_0\rangle. \tag{7}$$

We, therefore, define the Heisenberg picture operators by

$$X_H(t) = e^{iHt/\hbar}\, X_S\, e^{-iHt/\hbar}. \tag{8}$$

Equation of Motion In Heisenberg picture the state vector does not evolve with time. SO how do we describe the time development of a system? The answers is that in the Heisenberg picture the operators carry the entire time dependence. So for a point particle, the position operator, the momentum operator, in fact all dynamical variables become time dependent. This is parallel to the classical description the time evolution of the state is described by the position and momentum. The equations of motion are then the equations telling us how the a given dynamical variable will change with time. The equation of motion is easily derived from Eq.(8) and we compute

$$\frac{dX_H}{dt} = \frac{d}{dt}\Big[e^{iHt/\hbar}\,X_S\,e^{-iHt/\hbar}\Big] \tag{9}$$

$$= \frac{d}{dt}\Big(e^{iHt/\hbar}\Big)\,X_S\,e^{-iHt/\hbar} + e^{iHt/\hbar}\left(\frac{d}{dt}X_S\right)e^{-iHt/\hbar} + e^{iHt/\hbar}\,X_S\,\frac{d}{dt}\Big(e^{-iHt/\hbar}\Big) \tag{10}$$

$$= \frac{iH}{\hbar}e^{iHt/\hbar}\,X_S\,e^{-iHt/\hbar} + e^{iHt/\hbar}\left(\frac{\partial}{\partial t}X_S\right)e^{-iHt/\hbar} + e^{iHt/\hbar}\,X_S\;e^{-iHt/\hbar}\frac{-iH}{\hbar} \tag{11}$$

$$= \frac{i}{\hbar}HX_H + \frac{\partial}{\partial t}X_H - X_H\frac{i}{\hbar}H \tag{12}$$

Here we have used

$$\frac{d}{dt}e^{iHt/\hbar} = \left(\tfrac{iH}{\hbar}\right)e^{iHt/\hbar} = e^{iHt/\hbar}\left(\tfrac{iH}{\hbar}\right). \tag{13}$$

Thus we arrive at the final form of equations of motion in the Heisenberg picture

$$\boxed{\frac{dX_H}{dt} = \frac{\partial X_H}{\partial t} + \frac{1}{i\hbar}\big[X_H, H\big]_-.} \tag{14}$$

Recalling that the ($\frac{1}{i\hbar}\times$ commutator) has correspondence with the Poisson bracket, we have an obvious correspondence with the Poisson bracket form of equations of motion in classical mechanics.

The steps (9)- Eq.(12), leading to the final result (14), require some explanation and care as explained in Notes and Comments section at the end.

§3 Notes and comments

1. **Care needed with operators** The differentiation of an operator expression $e^{F(t)}$ with respect to time requires care. $F(t)$ may be an operator or a matrix. Both the following forms

$$\frac{de^{\lambda F(t)}}{dt} = \lambda\frac{dF(t)}{dt}e^{F(t)} = \lambda e^{F(t)}\frac{dF(t)}{dt} \tag{15}$$

   are valid if and only if the derivative $\frac{dF(t)}{dt}$ commutes with $e^{F(t)}$.

2. The result (15) has been derived under the assumption that the Hamiltonian H does not depend on time explicitly. When the Hamiltonian depends on time, the result (15) can be derives from Eq.(6) which defines the Heisenberg picture operators. Of course, in this case the integration of equation of motion gives a result which is more complicated than Eq.(8).

§4 Questions for you

A particle is described by the Hamiltonian

$$\hat{H} = \frac{\vec{p}^{\,2}}{2m} + V(\vec{x})$$

Find the equations of motion of the a particle using the Heisenberg representation of the position and momentum $\hat{x}(t)$ and $\hat{p}(t)$. Find $x(t), p(t)$, if initial ($t = 0$) conditions are given. Suppose the particle is in initial state ψ, what is the equation satisfied by the expectation value of x and p at a later time t? What does one expect classically?

T.V.Ramakrishnan

Consider the state $|\psi\rangle$ given by

$$|\psi\rangle = c_1|E_1\rangle + c_2|E_2\rangle.$$

Answer the following questions.

(a) Give condition(s) on E_1, E_2 so that $|\psi\rangle$ may be a stationary state for arbitrary c_1, c_2.

(b) Can the state represented by $|\psi\rangle$ be a stationary state when $E_1 \neq E_2$? If yes, give condition(s) on c_1, c_2.

(c) Let E_1, E_2, c_1, c_2 be such that $|\psi\rangle$ is **not** a stationary state. Compute the probability that a measurement of energy gives a value E_1 at time t. Does the probability vary time. WHY?

(d) Let E_1, E_2, c_1, c_2 be as in part (c). Now under what conditions that a dynamical variable X must obey so that the average of X remains constant?

§5 Notes and comments

[1] **Care needed with operators** The differentiation of an operator expression $e^{F(t)}$ with respect to time requires care. $F(t)$ may be an operator or a matrix. Both the following forms

$$\frac{de^{\lambda F(t)}}{dt} = \lambda \frac{dF(t)}{dt} e^{F(t)} = \lambda e^{F(t)} \frac{dF(t)}{dt} \tag{16}$$

are valid if and only if the derivative $\frac{dF(t)}{dt}$ commutes with $F(t)$.

[2] The equation

$$_H\langle\psi t|X_H(t)|\psi t\rangle_H = \, _S\langle\psi t|X_S|\psi t\rangle_S \tag{17}$$

which relates the Heisenberg picture operators and kets with the Schrodinger picture objects can be solved easily under the assumption that the Hamiltonian H does not depend on time explicitly. Assuming this to be the case, show that

$$X_H(t) = e^{iHt/\hbar} X_S \, e^{-iHt/\hbar}. \tag{18}$$

However, in the case when the Hamiltonian does depend on time, $H = H(t)$, the corresponding result is more complicated.