# QM-09 Lecture Notes
# Time evolution
# Time dependent Schrödinger equation
# Stationary states *

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**Abstract**

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A scheme to solve the time dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \tag{1}$$

is described. The solution will be presented in the form

$$|\psi t\rangle = U(t,t_0)|\psi t_0\rangle \tag{2}$$

Connection with stationary states is brought out clearly.

For our present discussion, it will be assumed that the Hamiltonian $\hat{H}$ does not depend on time explicitly. Let the state vector of system at initial time $t = 0$ be denoted by $|\psi_0\rangle$.

Since $\hat{H}$ is always assumed to be hermitian, its eigenvectors form an orthonormal complete set and we can expand the state vector at time $t$, $|\psi t\rangle$, in terms of the eigenvectors. Denoting the normalized eigenvectors by $|E_n\rangle$, we write

$$|\psi t\rangle = \sum_n c_n(t)|E_n\rangle. \tag{3}$$

where the constants $c_n(t)$ are to be determined. Substituting (3) in (1) we get

$$i\hbar\frac{d}{dt}\sum_n c_n(t)|E_n\rangle = \hat{H}|\psi t\rangle \tag{4}$$

$$i\sum_n \hbar\frac{dc_n(t)}{dt}|E_n\rangle = \sum_n c_n(t)\hat{H}|E_n\rangle \tag{5}$$

Taking scalar product with $|E_m\rangle$ and using orthonormal property of the eigenvectors $|E_n\rangle$, we get

$$i\hbar\frac{dc_m(t)}{dt} = E_m c_m(t). \tag{6}$$

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which is easily solved to give

$$c_m(t) = c_m(0)e^{-iE_m t/\hbar}. \tag{7}$$

Therefore, $|\psi t\rangle$,the solution of time dependent equation becomes

$$|\psi t\rangle = \sum_n c_m(0)e^{-iE_m t/\hbar}.|E_m\rangle. \tag{8}$$

The coefficients $c_m(0)$ are determined in terms of the state vector $|\psi_0\rangle$ at time $t = 0$ by setting time $t = 0$ in the above equation. This gives

$$|\psi_0\rangle = \sum_n c_n(0)|E_n\rangle. \tag{9}$$

The unknown coefficients $c_n(0)$ can now be computed; taking scalar product of Eq.(9) , with $|E_m\rangle$ we get

$$c_m(0) = \langle E_m|\psi_0\rangle. \tag{10}$$

Thus Eq.(8) and (10) give the solution of the time dependent Schrödinger equation as

$$\boxed{|\psi t\rangle = \sum_n c_n(0)\exp(-i\hbar E_n t)|E_n\rangle}. \tag{11}$$

Recalling the defintion of function of an operator, the above equation can be cast in the form

$$|\psi t\rangle = \exp(-iHt/\hbar)\sum_n c_n(0)|E_n\rangle \tag{12}$$

$$\therefore |\psi t\rangle = \exp(-iHt/\hbar)|\psi_0\rangle. \tag{13}$$

In general, if the state vector is know at time $t = t_0$, instead of time $t = 0$, the result Eq.(13) takes the form

$$|\psi t\rangle = \exp(-iH(t-t_0)/\hbar)\sum_n c_n(t_0)|E_n\rangle \tag{14}$$

$$= \exp(-iH(t-t_0)/\hbar)|\psi t_0\rangle. \tag{15}$$

The time evolution operator $U(t, t_0)$, of Eq.(2) , is therefore given by

$$\boxed{U(t, t_0) = \exp(-iH(t-t_0)/\hbar)}. \tag{16}$$

### Stationary states

Let us now consider time evolution of a system if it has a definite value of energy at an initial time $t_0$. The value of the energy then has to be one of the eigenvalues and the state vector will be the corresponding vector. So $|\psi t_0\rangle = |E_m\rangle$, then at time $t$ the system will be in the state given by

$$|\psi t\rangle = U(t, t_0)|E_m\rangle = \exp(-iE_m(t-t_0)/\hbar)|E_m\rangle. \tag{17}$$

It must be noted that the state vector at different times is equal to the initial state vector times a *numerical phase factor* $(\exp(-iE_m(t-t_0)/\hbar))$. Therefore, the vector at time $t$ represents the same state at all times. Such states are called **stationary states** because the state does not change with time. Every eigenvector of energy is a possible stationary state of a system. In such a state the average value of a dynamical variable, $\hat{X}$, not having time explicitly, is independent of time even if $\hat{X}$ does not commute with Hamiltonian. In fact the probabilities of finding a value on a measurement of the dynamical variable are independent of time.