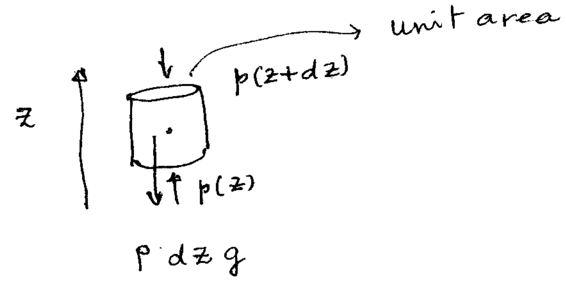
Problem sheet III SM Aug - Nov 2019



11. $p(z+dz) + \rho g\, dz = p(z)$

$$\frac{dp}{dz} = -\rho g$$

$$\frac{nm}{V} = \rho$$

We also have

$$pV = nRT = \frac{nm}{m} RT$$

$$p = \rho RT/m$$

$$\frac{dp}{dz} = -\frac{g\,p\,m}{RT}$$

Adiabatic $pV^{\gamma} = \text{Const}$

$$T^{\gamma} p^{1-\gamma} = \text{Const}$$

$$\gamma \frac{dT}{T} + (1-\gamma)\frac{dp}{p} = 0$$

$$\frac{dT}{dp} = \frac{\gamma - 1}{\gamma} \cdot \frac{T}{p}$$

$$\frac{dT}{dz} = \frac{dT}{dp} \cdot \frac{dp}{dz} = \frac{\gamma-1}{\gamma}\frac{T}{p}\left[-\frac{mgp}{RT}\right]$$

$$= -\left(\frac{\gamma-1}{\gamma R}\right) mg$$

$$= -\frac{0.4}{4 \times 8.3} \times 29 \times 9.8$$

$$= 9.8^{\circ}/\text{Km}.$$

12. $$dS = \frac{dU + p\,dV}{T}$$

$$\left.\frac{\partial S}{\partial V}\right)_T = \frac{1}{T}\left.\frac{\partial U}{\partial V}\right)_T + \frac{p}{T}$$

$$T\left.\frac{\partial S}{\partial T}\right)_V = \left.\frac{\partial U}{\partial T}\right)_V$$

$$\left.\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\left.\frac{\partial U}{\partial T}\right)_V$$

$$\frac{\partial^2 S}{\partial T\,\partial V} = -\frac{1}{T^2}\left.\frac{\partial U}{\partial V}\right)_T + \frac{1}{T}\frac{\partial^2 U}{\partial T\,\partial V} + \frac{1}{T}\frac{\partial p}{\partial T} - \frac{1}{T^2}p$$

$$= +\frac{1}{T}\frac{\partial^2 U}{\partial V\,\partial T}$$

$$p = f(V)\,T$$

$$\frac{\partial U}{\partial V} = p - T\frac{\partial p}{\partial T}$$

$$= f(V)\,T - T\,f(V) = 0,$$

13. $$\frac{\partial H}{\partial q} = -\frac{dp}{dt} = -ma \qquad \frac{\partial H}{\partial p} = \frac{dq}{dt} = \frac{p}{m}$$

$$p_A(t) = mat + p_A(0) \qquad q_A(t) - q_A(0) = \int_0^t \frac{p_A(t')}{m}\, dt'$$

$$q_A(t) - q_A = \int_0^t \left[at' + \frac{p_A(0)}{m}\right] dt' = \frac{at^2}{2} + \frac{p_A}{m} t$$

$$q_A(t) = q_A + \frac{at^2}{2} + \frac{p_A}{m} t$$

$$p_A(t) = mat + p_A.$$

$$q_B(t) = q_B + \frac{at^2}{2} + \frac{p_A}{m} t$$

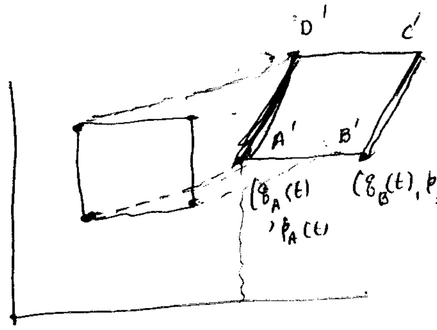
$$p_B(t) = mat + p_A = p_A(t)$$

$$q_C(t) = q_B + \frac{at^2}{2} + \frac{p_C}{m} t$$

$$p_C(t) = mat + p_C$$

$$q_D(t) = q_A + \frac{at^2}{2} + \frac{p_C}{m} t$$

$$p_D(t) = p_C + mat = p_C(t)$$



$$q_D(t) - q_A(t) = \frac{p_C - p_A}{m} t$$

$$p_D(t) - p_A(t) = p_C - p_A$$

Area of $A'B'C'D = (A'B')(p_C - p_A) = (q_B - q_A)(p_C - p_A)$

$= $ Area of $ABCD$.

14 $$N! = \int_0^\infty x^N e^{-x} dx = \int_0^\infty e^{N(\ln x - x/N)} dx \qquad g(x) = \ln x - \frac{x}{N}$$

max/min at $\frac{1}{x} = \frac{1}{N}$ $\quad x = N \qquad \frac{\partial^2 g}{\partial x^2} = -\frac{1}{x^2}\Big|_{x=N} = -\frac{1}{N^2}$ so max

$$N! = \int_{-N}^\infty e^{N\left[\ln(N+y) - \frac{N+y}{N}\right]} dy \qquad \ln(N+y) - 1 - y/N = \ln N - y^2/2N^2 - 1$$

$$= e^{N\ln N - N} \int_{-N}^{\infty} e^{-y^2/2N} dy \xrightarrow{N\to\infty} e^{N\ln N - N} \int_{-\infty}^{\infty} e^{-y^2/2N} dN$$

14 (Continued)

$$N! \approx e^{N\ln N - N}\sqrt{2\pi N} \qquad \int_{-\infty}^{\infty} e^{-z^2}\,dz = \sqrt{\pi}$$

$$\text{or} \quad \ln N! = N\ln N - N + \ln\sqrt{2\pi N}\,.$$

$$= N\ln N - N + \frac{1}{2}\ln(2\pi N)$$

15.

$$\rho = \frac{I + \vec{\sigma}\cdot\vec{P}}{2} \qquad \text{at } t = 0 \quad P_x = 0.4 \quad P_z = 0.6 \quad P_y = 0$$

$$i\hbar\frac{\partial\rho}{\partial t} = [H, \rho] = -\frac{\mu_0 B}{2}\left[\sigma_z,\ \sigma_x P_x + \sigma_y P_y + \sigma_z P_z\right]$$

$$= -\frac{\mu_0 B}{2}\left[2i\,\sigma_y P_x - 2i\,\sigma_x P_y\right]$$

$$\frac{\partial\rho}{\partial t} = \frac{\mu_0 B}{\hbar}\left[-P_x\sigma_y + P_y\sigma_x\right]$$

$$\frac{dP_x}{dt} = \frac{2\mu_0 B}{\hbar}P_y \qquad \frac{dP_z}{dt} = 0$$

$$\frac{dP_y}{dt} = -\frac{2\mu_0 B}{\hbar}P_x \qquad P_z = 0.6$$

$$\frac{\mu_0 B}{\hbar} = \omega$$

$$\frac{d^2P_x}{dt^2} = -4\omega^2 P_x \qquad P_x = A\cos 2\omega t + A'\sin 2\omega t$$

$$= 0.4\cos 2\omega t + A'\sin 2\omega t$$

$$\cancel{P_y(0) = 0 \quad A\sin 2\omega t + \ldots}$$

$$\frac{d^2P_y}{dt^2} = -4\omega^2 P_y$$

we have

$$\frac{dP_y}{dt} = -2\omega\left[(0.4)\cos 2\omega t + A'\sin 2\omega t\right]$$

$$P_y = +0.4\sin 2\omega t - A'(\cos 2\omega t - 1)$$

$$P_x = 0.4\cos 2\omega t + A'\sin 2\omega t$$

$$P_x^2 + P_y^2 = (0.4)^2 - 2A'\times 0.4\sin 2\omega t(\cos 2\omega t - 1) + A'^2(\cos 2\omega t - 1)^2$$

$$+ 2A'\times 0.4\cos 2\omega t\sin 2\omega t + A'^2\sin^2 2\omega t\,.$$

$$= (0.4)^2 - 0.8A'\sin 2\omega t + 2A'^2 - 2A'\cos 2\omega t$$

$\Rightarrow (A' = 0)$ see next sheet.

$$P_x = A \cos 2\omega t + B \sin 2\omega t$$

$$P_y = \frac{1}{2\omega} \frac{dP_x}{dt} = -A \sin 2\omega t + B \cos 2\omega t$$

At $t = 0$ $\quad P_y = 0 \quad B = 0$, $P_x = 0.4$

Thus $\quad P_z = 0.6 \quad P_x = 0.4 \cos 2\omega t \quad P_y = -0.4 \sin 2\omega t.$