Bundles Lessons in Vector Spaces

Part-III Linear Operators in Inner Product Spaces

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Lesson-1 Linear Operators in Inner Product Spaces

Lesson Overview

Syllabus Adjoint of an operator; Hermitian and Unitary operators.

Lesson Objecives To define the adjoint of of a linear operator in a finite Dimensional inner product space. To define hermitian and unitary operators; to discuss the properties of eigenvalues and eigenvectors of hermitian and unitary operators.

Prerequisites Understanding of the definition of vector spaces; concept of inner product. Linear functionals in an inner product space. and orthogonality.

References

1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).

2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).

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§1.1 Some preliminary results on linear operators

Theorem 1 *Let T be a linear operator in an inner product space. Let f and g be arbitrary vectors then*

$$(f, Tg) = (x_1, Tx_1) - (x_2, Tx_2) + i(x_3, Tx_3) - i(x_4, Tx_4)$$

where

$$x_1 = f + g; \quad x_2 = f - g; \quad x_3 = f - ig; \quad x_4 = f + ig$$

Proof follows by proceeding in a way similar to the proof of polarization identity. Use linearity of the operator T and expand the right hand side of the above identity to be proved.

Theorem 2 (When is a linear operator zero ?)

(1) If $(f, Tg) = 0$ holds for all f and g, then $T = 0$.

(2) If $(f, Tf) = 0$ is true for all $f \in \mathcal{V}$, then $T = 0$.

(3) If $(x_i, Tx_j) = 0$ holds for all elements x_i, x_j in a basis $\mathcal{X} = \{x_1, x_2, \ldots, x_N\}$ then $T = 0$.

Proof of (1) Given that $(g, Tf) = 0$ holds for all $g, f \in \mathcal{V}$. Therefore, we take $g = Tf$. This gives $(Tf, Tf) = 0$ which in turn implies $Tf = 0$ for all $f \in \mathcal{V}$. Therefore, $T = 0$.

Proof of (2) Using linearity of T we have already proved the result that

$$4(f, Tg) = (x_1, Tx_1) - (x_2, Tx_2) - i(x_3, Tx_3) + i(x_4, Tx_4)$$

where

$$x_1 = f + g; x_2 = f - g; x_3 = f - ig; x_4 = f + ig$$

Since $(f, Tf) = 0$ for all vectors $f \in \mathcal{V}$, the right hand side is zero. Hence we get $(f, Tg) = 0$ for all f, g in the vector space . Hence using part (2) we get $T = 0$.

Proof of (3) Since an arbitrary vector f can be written as a linear combination, $f = \sum \alpha_k x_k$, we can prove (2) by using linearity of T. Hence the result $T = 0 follows.$

§1.2 Adjoint of an Operator

Representation theorem for linear functionals It can be proved that for every linear functional $\psi : f \longrightarrow \psi(f)$ on a complex inner product space $\mathcal{V}$ of finite dimension, there exists a vector $g \in \mathcal{V}$ such that $<< Proof?? >>$

$$\psi(f) = (g, f) \tag{1}$$

Definition 1

Given a linear operator T on an inner product space of finite dimension now we shall define **adjoint** of T, to be denoted by $T^\dagger$. The adjoint $T^\dagger$ will be defined once its action on an arbitrary vector f is specified. The functional ϕ, defined by

$$\phi : g \to \phi(g) = (f, Tg), \tag{2}$$

is a linear functional on $\mathcal{V}$, and hence there exists a vector unique vector $h \in \mathcal{V}$ such that

$$(h, g) = \phi(g) = (f, Tg) \tag{3}$$

We, then, define $T^\dagger f = h$. Thus the operator $T^\dagger$ has the property

$$(f, Tg) = (T^\dagger f, g), \qquad \forall f, g \in \mathcal{V}. \tag{4}$$

Remark If we find $(f, Tg) = (Xf, g)$ holds for all f and g we can conclude immediately that $X = T^\dagger$. WHY ? ¡¡ Proof ?! ¿¿

Properties of the adjoint

1. $A^\dagger$, the adjoint of a linear operator is again a linear operator.
2. $(\alpha A)^\dagger = \alpha^* A^\dagger$
3. $(A + B)^\dagger = A^\dagger + B^\dagger$
4. $(AB)^\dagger = B^\dagger A^\dagger$
5. If A is invertible, $A^\dagger$ is invertible and $(A^\dagger)^{-1} = (A^{-1})^\dagger$.

§1.3 Hermitian Operator

Definition 2 *An operator A is* **hermitian** *if $A^\dagger = A$.*

Theorem 3 (When is an operator hermitian ?) *Each of the following two statements give condition for hermiticity of an operator.*

(H1) An operator T is hermitian if and only if $(Tg, f) = (g, Tf)$ holds for all $f, g \in \mathcal{V}$.

(H2) In a finite dimensional vector space an operator T is self adjoint if and only if (f, Tf) is real $\forall f \in \mathcal{V}$.

Proof of (H1) : Let T be a hermitian operator. Using the definition of adjoint we have

$$(g, Tf) = (T^\dagger g, f)$$

or

$$(g, Tf) = (Tg, f) \qquad (\because T = T^\dagger)$$

Let $(g, Tf) = (Tg, f)$ for all f and g in the vector space. Then we get

$$(g, Tf) = (Tg, f) \qquad \text{(given)} \tag{5}$$
$$(g, Tf) = (g, T^\dagger f) \quad \text{(Use def of } T^\dagger) \tag{6}$$
$$(g, (T - T^\dagger)f) = 0 \tag{7}$$

holds $\forall g$ and f. Select $g = (T - T^\dagger)f$. This gives $\|(T - T^\dagger)f\| = 0$. Therefore,

$$(T - T^\dagger)f = 0 \qquad \forall f \in \mathcal{V}. \tag{8}$$

Hence $T = T^\dagger$

Proof of (H2) : Let (f, Tf) be real. Then

$$\begin{aligned}(f, Tf) &= (f, Tf)^* \qquad \text{given} && (9)\\ &= (Tf, f) \qquad \text{(property of inner product)} && (10)\\ &= (f, T^\dagger f) \qquad \text{(def of adjoint)} && (11)\end{aligned}$$

Thus $(f, Tf) = (f, T^\dagger f)$ holds $\forall f \in \mathcal{V}$. This implies $(f, (T - T^\dagger)f) = 0$, hence $T = T^\dagger$. Therefore, T is hermitian.

Theorem 4 *If X is any operator we may write, $X = A + iB$, where A and B are hermitian operators.*

The proof is easy. We write $A = (X + X^\dagger)/2; \qquad B = (X - X^\dagger)/2i$ It is straight forward to verify that A and B are hermitian and that $X = A + B$.

)‖(**Short Examples 1** *Consider the Hilbert space $\mathcal{L}^2(-\infty, \infty)$ of square integrable functions.*

(1a) Let $\hat{X}$ be defined as

$$\hat{X}\psi(x) = e^{ikx}\psi(x), \quad k \in \mathbb{R}$$

then

$$\hat{X}^\dagger\psi(x) = e^{-ikx}\psi(x), \quad k \in \mathbb{R}$$

The operator $\hat{X}$ is not hermitian.

(1b) The parity operator $\hat{P}$ defined by

$$\hat{P}\psi(\vec{x}) = \psi(-vecx)$$

is hermitian, $\hat{P}^\dagger = \hat{P}$ because

$$(P\psi(x), \phi(x)) = (\psi(x), P\phi(x).$$

(1c) The adjoint of translation operator $\hat{T}$ defined by

$$\hat{T}\psi(x) = \psi(x + a)$$

is given by .

$$\hat{T}^\dagger\psi(x) = \psi(x - a)$$

(1d) The translation operator $\hat{T}$ defined above is a not hermitian operator, $T^\dagger \neq T$.

Theorem 5 (Eigenvalues and Eigenvectors of Hermitian Operators) *Two important properties of hermitian operators are given below.*

(E1) The eigenvalues of a hermitian operators are real.

(E2) The eigenvectors of a hermitian operator corresponding to two distinct eigenvalues are orthogonal.

Proof of (E1) : Let λ be an eigenvalue and f be eigenvector of T with $Tf = \lambda f$. Since T is a hermitian operator we have

$$(x, Ty) = (Tx, y), \qquad \forall\, x, y \in \mathcal{V}. \tag{12}$$

Therefore setting $x = y = f$ in (12), we get $(f, Tf) = (Tf, f)$ we get

$$(Tf, f) = (f, Tf) \tag{13}$$

$$\Rightarrow \qquad (\lambda f, f) = (f, \lambda f) \tag{14}$$

$$\therefore \qquad (\lambda^* - \lambda)(f, f) = 0 \tag{15}$$

As $f \neq 0, (f, f) \neq 0$ and hence we must have $\lambda^* - \lambda = 0$ Therefore the eigenvalues of a hermitian operator are real.

Proof of (E2) :To prove that two eigenvectors corresponding to a different eigenvalues are orthogonal. Let $Tf = \lambda f$ and $Tg = \mu g$. and T be a hermitian operator $T\dagger = T$ and $\lambda \neq \mu$. Then proceeding as in proof of (E1)

$$(f, Tg) = (Tf, g) \qquad (\text{Since } T^\dagger = T)$$

We have

$$(f, \mu g) = (\lambda f, g)$$

or

$$\mu(f, g) = \lambda^*(f, g) = \lambda(f, g)$$

because the eigenvalues λ, μ are real. For $\lambda \neq \mu$ the above equation implies that $(f, g) = 0$. Hence f and g are orthogonal.

§1.4 Unitary operator

Definition 3 *A linear operator U is called* **unitary** *if $U^\dagger = U^{-1}$. In case of a finite dimensional vector space, it is equivalent to demanding*

$$UU^\dagger = I, \qquad (\text{ or } U^\dagger U = I).$$

Theorem 6 (When is an Operator Unitary ?) *In a finite dimensional vector space, $\mathcal{V}$, the following conditions on an operator X are equivalent.*

(U1) X *is unitary.*

(U2) $(Xf, Xg) = (f, g) \qquad \forall f, g \in \mathcal{V}$

(U3) $\|Xf\| = \|f\| \qquad \forall f \in \mathcal{V}$.

Proof of $(U1) \Rightarrow (U2)$: $(Xf, Xg) = (f, X^\dagger Xg) = (f, g)$

Proof of $(U2) \Rightarrow (U3)$: (U3) follows from (U2) by setting $g = f$ in (U2).

Proof of $(U3) \Rightarrow (U1)$: Given that $\|Xf\| = \|f\| \quad \forall f \in \mathcal{V}$ we have $(Xf, Xf) = (f, f)$. This in turn gives $(f, X^\dagger Xf) = (f, f)$ or

$$(f, (X^\dagger X - I)f) = 0 \forall f \in \mathcal{V}$$

Thus $(X \dagger X - I) = 0$. This means that $X^\dagger X = I$. In finite dimensional spaces we then have the result that X is unitary.

)|||(**Short Examples 2** *Consider the Hilbert space $\mathcal{L}^2(-\infty, \infty)$ of square integrable functions. Consider again the operators considered in*)|||(*Short Examples 1 above..*

(2a) Let $\hat{X}$ be defined as

$$\hat{X}\psi(x) = e^{ikx}\psi(x), \quad k \in \mathbb{R}$$

then

$$\hat{X}^\dagger\psi(x) = e^{-ikx}\psi(x), \quad k \in \mathbb{R}$$

The operator $\hat{X}$ is unitary, because $\hat{X}\psi(x), \hat{X}\phi(x)) = (\psi, \phi$

(2b) The parity operator $\hat{P}$ defined by

$$\hat{P}\psi(\vec{x}) = \psi(-vecx)$$

is unitary. $\hat{P}^\dagger = \hat{P}$ and therefore $(P^2\psi(x) = P\psi(-x) = \psi(x)$. Thus

$$\hat{P}^2 = I \Rightarrow \hat{P}^\dagger\hat{P} = I = \hat{P}\hat{P}^\dagger.$$

(2c) The adjoint of translation operator $\hat{T}$ defined by

$$\hat{T}\psi(x) = \psi(x + a)$$

is given by .

$$\hat{T}^\dagger\psi(x) = \psi(x - a)$$

(2d) The translation operator $\hat{T}$ defined above is a unitary operator, $T^\dagger \neq T$.

Theorem 7 (Eigenvalues and Eigenvectors of Unitary Operators) *If λ as an eigenvalue of a unitary operator, then*

(E3) Absolute value of λ is unity, $|\lambda| = 1$. It can be equivalently written in several forms such as $\lambda^*\lambda = 1$, or $\lambda^* = 1/\lambda$, or $\lambda = e^{i\alpha}$ with $\alpha \in \mathbb{R}$.

(E4) The eigenvectors of a unitary operator corresponding to two distinct eigenvalues are orthogonal.

Proof of (E3) *: Let U be a unitary operator having λ as an eigenvalue and f as an eigenvector.*

$$\begin{aligned} Uf &= \lambda f && (16) \\ \Rightarrow \qquad (Uf, Uf) &= (f, f), \quad (\text{ since } U \text{ is unitary }) && (17) \\ \Rightarrow (\lambda f, \lambda f) &= (f, f) && (18) \\ \Rightarrow \lambda * \lambda(f, f) &= (f, f) && (19) \\ \Rightarrow \big(|\lambda|^2 - 1\big)(f, f) &= 0 && (20) \\ \Rightarrow |\lambda|^2 &= 1, \qquad \because (f, f) \neq 0 && (21) \end{aligned}$$

Therefore, $|\lambda| = 1$. This means that λ is phase and $\lambda = \exp(i\alpha)$.

Proof of (E4) : *Let λ and μ be two distinct eigenvalues of a unitary operator U and let f and g be the corresponding eigenvectors. Thus*

$$Uf = \lambda f, \qquad Ug = \mu g, \qquad \text{and } \lambda \neq \mu. \tag{22}$$

Since U is unitary

$$\begin{aligned} (Uf, Ug) &= (f, g) & (23)\\ \Rightarrow (\lambda f, \mu g) &= (f, g) & (24)\\ \Rightarrow (\lambda^* \mu)(f, g) &= (f, g) & (25) \end{aligned}$$

Since $|\lambda| = 1$, we have $\lambda^*\lambda = 1$, or $\lambda^* = 1/\lambda$. The above equation then gives

$$\begin{aligned} [(\mu/\lambda) - 1](f, g) &= 0 & (26)\\ \therefore (f, g) &= 0, \qquad \because \mu \neq \lambda \text{ and } (\mu/\lambda - 1) \neq 0. & (27) \end{aligned}$$

)|||(***Short Examples 3*** *The matrix*

$$S = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is real orthogonal and hence unitary. Find its eigenvalues and verify that their moduli are unity.

Lesson-2 Spectral Theorem

Lesson Overview

Syllabus Normal operator; spectral theorem.

Lesson Objecives To define normal operator, and discuss its properties. The spectral theorem for normal operators is briefly explained.

Prerequisites Understanding of the definition of vector spaces; concept of inner product. Linear operators in an inner product space. Commutator of two linear operators.

References

1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).

2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).

§2.1 Normal Operators

Two important references for this lecture are [**?**], and [**?**].

Definition 1 *An operator on an inner product space* $\mathcal{V}$ *is called a* **normal operator** *if it commutes with its adjoint.*

Proposition An operator T on an inner product space $\mathcal{V}$ is normal if and only if $\|Tx\| = \|T^\dagger x\|$ for all $x \in \mathcal{V}$.

To prove this consider

$$(Tx, Tx) = (T^\dagger x, T^\dagger x) \iff (x, T^\dagger Tx) = (xTT^\dagger x) \tag{28}$$

$$\iff (x, (TT^\dagger - T^\dagger T)) \tag{29}$$

$$\iff TT^\dagger - T^\dagger T = 0 \tag{30}$$

This completes the required proof.

Theorem 8 *Let* T *be a normal operator on an inner product space. Then* u *is eigenvector of* T *with eigenvalue* λ *if and only if* u *is an eigenvector of* $T^\dagger$ *with eigenvalue* λ^**.*

Proof

Let I be the identity operator, and operator T be a normal operator. Then $X \equiv T - \lambda I)$ is normal for all $\lambda \in \mathbb{C}$, because $X^\dagger = T^\dagger - \lambda^* I$ and

$$\left[X, X^\dagger\right] = \left[T - \lambda I, T^\dagger - \lambda^* I\right] \tag{31}$$

$$= \left[T, T^\dagger\right] - \lambda\left[I, T^\dagger\right] - \lambda\left[I, T^\dagger\right] + |\lambda|^2\left[I, I\right] \tag{32}$$

$$= \left[T, T^\dagger\right] = 0. \tag{33}$$

That u is and eigenvector of T implies $Xu = 0 \Rightarrow \|Xu\| = 0$. By the previous theorem $\|X^\dagger u\| = 0$.Thus

$$\|X^\dagger u\| = 0 \Longrightarrow X^\dagger u = 0 \Longrightarrow (X^\dagger - \lambda^* I)u = 0 \Longrightarrow X^\dagger u = \lambda^* u. \tag{34}$$

This completes the required proof.

Definition 2 (Normal Operator) *An operator* A *is called normal if it commute with its adjoint*

$$AA^\dagger - A^\dagger A = 0.$$

The unitary and hermitian operators are subset of class of all normal operators. The normal operators shares orthogonality and completeness properties of eigenvectors with unitary and hermitian operators. The spectral theorem holds for the normal operators. For more details see references.

Theorem 9 *If* A *is normal, then a necessary and sufficient condition that* x *be an eignevector of* A *is that it be a an eigenvector of* $A^\dagger$*; if* $Ax = \lambda x$ *then* $A^\dagger x = \lambda^* x$*.*

If A is normal, then we have

$$\|Ax\| = (Ax, Ax) = (A^\dagger Ax, x) = (AA^\dagger x, x) = (A^\dagger x, A^\dagger x) = \|A^\dagger x\|. \quad (35)$$

For every complex λ, normal operator A, $A - \lambda$ is also normal. Then we have

$$\|(A^\dagger - \lambda^*)x\| = \|(A - \lambda)x\| = 0 \quad (36)$$

This proves that x is an eigenvector of A with eigenvalue λ if and only if x is an eigenvector of $A^\dagger$ with eigenvalue λ^*.

Theorem 10 *If A is normal then the eigenvectors belonging to distinct eigenvalues are orthogonal.*

Let x_1, x_2 be the eigenvectors of A with eigenvalues λ_1, Λ_2, then

$$Ax_1 = \lambda x_1 \Rightarrow (x_2, Ax_1) = \lambda_1(x_1, x_1) \quad (37)$$

Also we have

$$Ax_2 = \lambda_2 x_2 \quad \Rightarrow \quad A^\dagger x_2 = \lambda_2^* x_2 \quad (38)$$

$$\therefore \quad (A^\dagger x_2, x_1) \quad = \quad (\lambda_2^* x_2, x_1) = \lambda_2(x_2, x_1) \quad (39)$$

Subtracting (37) and (39) we get

$$(x_2, Ax_1) - (A^\dagger x_2, x_1) = (\lambda_1 - \lambda_2)(x_2, x_1). \quad (40)$$

The left hand side is zero, (definition of adjoint). Therefore $(x_2, x_1) = 0$ if $\lambda_1 \neq \lambda_2$.

§2.2 Spectral Theorem

It is an important question to ask when will the set of all eignevectors of an form a basis. A large class of operators is the class of normal operators containing the set of all hermitian and the set of all unitary operators. The eigenvectors of a normal operator shares orthogonality property with hermitian and unitary operators. So we begin with a result on normal operators.

Theorem 11 *If A is normal, λ is an eigenvalue of A, and $\mathcal{M}$ is the set of all solutions of $Ax = \lambda x$, the both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under A.*

If a vector x belongs to both $\mathcal{M}$ and $\mathcal{M}^\perp$, it must be orthogonal to itself and therefore it must be the null vector, $x = 0$.

Theorem 12 (Completeness of eigenvectors) *For a normal operator in a finite dimensional vector space the eigenvectors span the whole space.*

Let A be a normal operator and $\mathcal{M}$ be the subspace spanned by its eigenvectors, and let E be the projection operator onto $\mathcal{M}$. Then $\mathcal{M}^{\perp}$ is invariant under A.

To prove by contradiction we assume that $\mathcal{M}$ is not whole space. Then $\mathcal{M}^{\perp}$ is a finite dimensional subspace with dimension greater than zero. Consider $A(1-E)$ as an operator on $\mathcal{M}^{\perp}$. Since every operator on a finite dimensional vector space has at least one eigenvalue, there exists a number α and a nonzero vector y such that $A(1-E)y = \alpha y$. Then we would have

$$Ay = AEy + A(1-E)y = A(1-E)y = \alpha y. \tag{41}$$

So y is an eigenvector of A and should belong to $\mathcal{M}$. Thus y is a nonzero vector common to both $\mathcal{M}$ and $\mathcal{M}^{\perp}$. This contradicts the fact that the only vector common to both $\mathcal{M}$ and $Mca^{\perp}$ is the null vector.

The above result can be stated in a form that can be generalized for infinite dimensional vector spaces.

Theorem 13 (Spectral theorem) *To every normal linear transformation A on a finite dimensional vector space with inner product there correspond numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ and perpendicular projections $P_1, P_2, \ldots, P_r$ where r is strictly positive integer, not greater than the dimension of the vector space) so that*

1. *the λ_j are pairwise distinct*
2. *the projections P_j are pairwise orthogonal and different from zero.*
3. $\sum_i P_I = I$
4. $\sum_i \lambda_i P_i = A$

For a proof we refer to the books by Jordan and by S. Hassani.

Definition 3 *Using the spectral theorem we can now define functions of operators. If A is a normal operator as before, an operator function $F(A)$ is defined by*

$$F(A) = \sum_i F(\lambda_i) P_i \tag{42}$$

The relationship of normal operators with hermitian operators, unitary operators etc is give by the following theorem.

Theorem 14 *A normal operator on a finite dimensional complex vector space is*

- *hermitian if and only if all its eigenvalues are real;*
- *positive if and only if all its eigenvalues are positive;*
- *strictly positive if and only if all its eignevalues are strictly positive;*
- *unitary of all its eigenvalues have absolute value 1;*

- *invertible if and only if all its eigenvalues are different from zero;*
- *idempotent if and only if all its eigenvalues are equal to zero or 1.*

For more details see Halmos. For spectral theorem for operators in infinite dimensional vector spaces see Jordan [1], Sadri Hassani[2], Halmos[3].

References

[1] T. F. Jordan, *Linear operators for quantum mechanics*, John Wiley and Sons, New York (1969).

[2] S. Hassani, *Mathematical Physics*, Springer (1998).

[3] P.R. Halmos, *Finite Dimensional Vector Spaces*, Springer Verlag (1974).