Bundled Lessons in Vector Spaces

Bundle-1 Basic Concepts

A. K. Kapoor
http://0space.org/users/kapoor
akkapoor@cmi.ac.in; akkhcu@gmail.com

Contents

1 Groups and Fields **2**
§1.1 Recall :: Binary Operation 2
§1.2 Groups 3
§1.3 Fields 4

2 Vector Spaces and Subspaces **5**
§2.1 Vector Spaces 5
§2.2 Subspaces 7

3 Linear Independence and Basis **9**
§3.1 Linear independence 9
§3.2 Basis and dimension 10

4 Linear Functional, Dual Vector Space **12**
§4.1 Linear functional 12
§4.2 Dual vector space 13

5 New Vector Spaces from Old **15**
§5.1 Sum of subspaces 15
§5.2 Quotient space 16
§5.3 Tensor product of vector spaces 18

Lesson-1 Groups and Fields

Lesson Overview

Syllabus Groups, Fields

Lesson Objectives You will learn definition of group and field with some examples and counter examples.

Prerequisite Basic set theory, binary operation

References

1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).

2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).

---

§1.1 Recall :: Binary Operation

We will not construct the real numbers from rationals. We take real numbers as undefined objects which satisfy certain axioms. Starting from these axioms, all familiar properties can be proved.

The axioms for real number system come in three groups.

1. The Field axioms

2. The Order axioms

3. The Completeness axiom, or the least upper bound axiom.

We assume that the set $\mathbb{R}$ of real numbers is given to us and also given to us is a set $\mathbb{P} \in \mathbb{R}$ of positive reals. We also assume that two binary operations $+$ and $\cdot$ are defined. We assume that $\mathbb{P}, \mathbb{R}$ $+$ and $\cdot$ satisfy the following relations.

THE FIELD AXIOMS For all $x, y, z \in \mathbb{R}$, we have

(A1) $x + y = y + x$;

(A2) $(x + y) + z = x + (y + z)$;

(A3) $\exists 0 \in \mathbb{R}$ s.t. $x + 0 = x, \forall x \in \mathbb{R}$;

(A4) For each $x \in \mathbb{R}$, there exists a $v \in \mathbb{R}$ s.t. $x + v = 0$.
Such a v is called 'additive inverse' of x and is denoted by $-x$;

(A5) $x \cdot y = y \cdot x, \qquad \forall x, y \in \mathbb{R}$;

(A6) $(x \cdot y)z = x(y \cdot z), \qquad \forall x, y, z \in \mathbb{R}$

(A7) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \neq 0$ and $x.1 = x, \quad \forall x \in \mathbb{R}$;

(A8) $\forall x \in \mathbb{R}, \quad x \neq 0,$ there exists $w \in \mathbb{R}$ s.t. $x\,w = 1$;
w will be called multiplicative inverse of x;

(A9) Distributive law: $x(y + z) = xy + zx$.

B. AXIOM OF ORDER The subset $\mathbb{P}$ of positive real numbers satisfies the following axioms.

(B1) $x, y \in \mathbb{P} \Rightarrow x + y \in \mathbb{P}$;

(B2) $x, y \in \mathbb{P} \Rightarrow x.y \in \mathbb{P}$;

(B3) $x \in \mathbb{P} \Rightarrow -x \notin \mathbb{P}$;

(B4) $x \in \mathbb{R} \Rightarrow x = 0$, or $x =\notin \mathbb{P}$, or $x \in \mathbb{P}$
$i.e.$ $\mathbb{R} = -\mathbb{P} \cup \{0\} \cup \mathbb{P}$, where $-\mathbb{P}$ is the set $\{x : -x \in \mathbb{P}\}$.

Using the above axiom, the familiar properties of order relation $<$ can be proved, if we define

$$x < y \text{ to mean } y - x \in \mathbb{P}.$$

Thus we have
(B1) $\Rightarrow$ $x < y$ and $z < w \Rightarrow x + z < y + w$;
(B2) $\Rightarrow$ $0 < x < y$ and $0 < z < w \Rightarrow xz < yw$
(B4) $\Rightarrow$ If $x \in \mathbb{R}, y \in \mathbb{R}$, only one of the following holds .
$x < y$, or $x = y$, or $y < x$.
The last axiom given below the most important one.

C. COMPLETENESS AXIOM or THE LEAST UPPER BOUND AXIOM Every nonempty set of real numbers which has an upper bound has a least upper bound.

§1.2 Groups

Definition 1 *To every ordered pair $\langle a, b\rangle$ of elements of a set $\mathcal{X}$ a* **binary operation** *assigns an element, denoted by $a * b$, of the set $\mathcal{X}$. For a binary operation to be a valid one it must be defined for all pairs and the $a * b$ must belong to the set and the result of binary operation must be unique.*

Definition 2 *A* **group** *is a pair $\langle \mathcal{G}, *\rangle$ with a binary operation $*$ defined on a set $\mathcal{G}$ such that the following properties.*

(G-1) Associative property : $a * (b * c) = (a * b) * c \qquad \forall\ a, b, c \in \mathcal{G}$

(G-2) Existence of identity : $\exists$ an element $e \in \mathcal{G}$ such that

$$e * a = a * e = a \qquad \forall\ a \in \mathcal{G}.$$

(G-3) Existence of inverse : $\forall\ a \in \mathcal{G}$ there exists an element a' such that

$$a * a' = a' * a = e$$

Examples Of Groups

1. The set of all real numbers $\mathbb{R}$ forms a group with addition as the binary operation.

2. The set of all complex numbers $\mathbb{C}$ is a group with addition as group operation.

3. The set of all positive, non-zero, real numbers $\mathbb{R}^+$ is a group with respect to multiplication as group operation.

4. The set of all $N \times N$ real (or complex) matrices form a group under matrix addition.

5. The group of all $N \times N$ real (or complex) matrices with determinant $\neq 0$ form a group under the matrix multiplication.

§1.3 Fields

Definition 3 *A* **field** $\mathcal{F}$ *is a triple* $\langle \mathcal{F}, +, \cdot \rangle$, *where,* $\cdot$ *and* $+$ *are two binary operations defined on a set* $\mathcal{F}$ *such that the axioms (F-I) to (F-III), given below, are satisfied. The elements of the field will be called* **scalars** *and will be denoted by greek letters* $\alpha, \beta, \gamma, \ldots$.

(F-1) To every pair α, β the scalar $\alpha + \beta$ is called the **sum** of α, β which satsifies the following axioms $\forall \alpha, \beta, \gamma \in \mathcal{F}$

(i) $\alpha + \beta = \beta + \alpha$

(ii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

(iii) $\exists$ a unique scalar 0 such that $0 + \alpha = \alpha = \alpha + 0$

(iv) $\forall \alpha \in \mathcal{F}\ \exists$ a unique scalar $(-\alpha) \in \mathcal{F}$ we have $\alpha + (-\alpha) = 0$

These properties imply that $\mathcal{F}$ is a group with $+$ as binary operation.

(F-2) The scalar $\alpha \cdot \beta$ will be called the **product** of α, β and has the following properties.

(i) Commutative Property : $\alpha \cdot \beta = \beta \cdot \alpha$

(ii) Associative Property : $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

(iii) Existence of multiplicative identity : $\exists$ a unique scalar 1 such that

$$\alpha.1 = 1.\alpha = \alpha$$

(iv) $\forall \alpha \neq 0\ \exists$ a scalar denoted by α^{-1} such that $\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1$

(F-3) The sum and the product obey the distributive property :

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

Examples Of Fields

1. Set of all rational numbers $\mathbb{Q}$ is a field with usual addition and multiplication as the two binary operations.

2. Set of all real numbers $\mathbb{R}$ is a field with usual addition and multiplication as the two binary operations.

3. Set of all complex numbers $\mathbb{C}$ is a field with usual addition and multiplication as the two binary operations.

4. The set $\mathbb{Z}^+$ of all positive integers is not a field with the usual addition and multiplication as two binary operations (give all possible reasons).

5. The set $\mathbb{Z}$ of all integers is not a field with the usual addition and multiplication. (Give one reason).

Lesson-2 Vector Spaces and Subspaces

Lesson Overview

Syllabus Vector Spaces; Subspace of a vector space

Prerequisites Basic set theory; Groups and fields

Lesson Objectives To define vector space and subspace; to illustrate the definitions with examples and counter examples.

References

1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).

2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).

---

§2.1 Vector Spaces

Definition 4 *Let $\mathcal{F}$ be a field and $+$ be a binary operation defined on a set $\mathcal{V}$. The triple $\langle \mathcal{V}, +, \mathbb{F} \rangle$ is a* **vector space** *on a field $\mathcal{F}$ if the following properties are satisfied.*

(V-1) To every pair of vectors $f, g \in \mathcal{V}$, there corresponds a vector $f + g \in \mathcal{V}$ called the sum of f and g such that

(i) $f + g = g + f \qquad \forall\ f, g \in \mathcal{V}$

(ii) $f + (g + h) = (f + g) + h \qquad \forall\ f, g, h \in \mathcal{V}$

(iii) $\exists$ a unique vector $0 \in \mathcal{V}$ such that

$$f + 0 = f \qquad \forall\ f \in \mathcal{V}$$

(iv) To every vector $f \in \mathcal{V}$, there corresponds a vector $-f \in \mathcal{V}$ such that

$$f + (-f) = 0$$

(V-2) $\forall\ \alpha \in \mathcal{F}$ and $f \in \mathcal{V}$ there corresponds a unique vector $\alpha f \in \mathcal{V}$ such that

$$\alpha(\beta f) = (\alpha\beta)f \qquad \forall\ \alpha, \beta \in \mathcal{F}$$

and

$$1.f = f \qquad \forall f \in \mathcal{V}$$

(V-3) $\forall \alpha, \beta \in \mathcal{F}$ and $\forall f, g \in \mathcal{V}$ we have

$$(\alpha + \beta)f = \alpha f + \beta f$$

and

$$\alpha(f + g) = \alpha f + \alpha g$$

Examples Of Vector Spaces

(I)
1. Every field $\mathcal{F}$ is also a vector space over $\mathcal{F}$ as field of scalars. Thus we have the following important special examples of vector spaces.
2. Set of all complex numbers $\mathbb{C}$ is a complex vector space with $\mathbb{C}$ as the field of scalars.
3. Set of all real numbers $\mathbb{R}$ is a real vector space with $\mathbb{R}$ as the field of scalars.
4. Set of all rational numbers $\mathbb{Q}$ is a rational vector space with $\mathbb{Q}$ as the field of scalars.

(II) Set of all n-tuples $(\alpha_1, \alpha_2, ..., \alpha_n)$ where $\alpha_k \in \mathcal{F}$ is denoted by $\mathcal{F}^n$. This set is vector space with $\mathcal{F}$ as field of scalars. Thus

1. $\mathbb{C}^n$ is a complex vector space over $\mathbb{C}$ as the field of scalars.
2. $\mathbb{R}^n$ is a real vector space over $\mathbb{R}$ as the field of scalars.
3. $\mathbb{Q}^n$ is a rational vector space over $\mathbb{Q}$ as the field of scalars.

(III)
1. All polynomials in a variable t, with coefficients in any field $\mathcal{F}$ is vector space $\mathcal{P}$.

$$\mathcal{P} = \{p(t)|p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t + ... + \alpha_n t^n + ...\text{and } \alpha_j \in \mathcal{F}\}$$

Here $\mathcal{F}$ can be any of the fields such as $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \ldots$

2. Consider the set $\mathcal{P}$ of all polynomials in a variable t, with coefficients in any field $\mathcal{F}$ and consider the subset $\mathcal{P}_N$ consisting of all polynomials of degree $\leq N$. Then $\mathcal{P}_N$ is a vector space.

(IV)
1. Let $\mathcal{F}$ be set of all functions defined on an interval $[a, b]$ and having complex values. With any one of the fields $\mathbb{C}, \mathbb{R}$, or $\mathcal{Q}$, $\mathcal{F}$ is a vector space.
2. Let $\mathcal{F}$ be as in (IV-1) and $\mathcal{C}^{(0)}$ be the subset of all continuous functions. Then $C^{(0)}$ is a vector space.
3. Let $\mathcal{F}$ be as in (IV-1) and $C^{(r)}$ be the subset of all functions for which $r-$ derivatives exist and are continuous on $[a, b]$. The $C^{(r)}$ is a vector space.
4. Let $\mathcal{C}^{(0)}$ be as in (IV-2). Let $\mathcal{S}$ be a subset of $\mathcal{C}^{(0)}$ consisting of those functions which vanish at a given point x_0.Then $\mathcal{S}$ is vector space. In general, if one can takes all functions which vanish at $x_1, x_2, \ldots, x_n$ then also we get a vector space.

(V) Let $\mathbb{M}_N$ be the set of all $N \times N$ matrices whose element are scalars from a field $\mathcal{F}$. With standard matrix addition as vector addition $\mathbb{M}_N$ is a vector space over the same field F

(VI) The set of all functions f on an interval $[a, b]$, for which $\int_a^b |f(x)|^p \, dx$ is finite, is a vector space denoted by $\mathcal{L}^p[a, b]$. That addition of two functions in $\mathcal{L}^p[a,b]$ gives back a function in the same space will not be proved here. The space $\mathcal{L}^p[a,b]$, for $p=2$, is the set of all square integrable functions on the interval $[a,b]$.

(VII) The set of all infinite sequences $(\alpha_1, \alpha_2, \ldots, ..)$, such that the infinite series

$$\sum_{k=1}^{\infty} |\alpha_k|^p$$

converges, is a vector space denoted by ℓ^p. That the sum of two sequences, $\alpha, \beta \in \ell^p$ is also in ℓ^p, space requires a proof which will not be given here.

(VIII) A set $\{0\}$, consisting of only one element, the null vector, is a vector space over any field.

§2.2 Subspaces

Definition 5 *Let $\mathcal{V}$ be a vector space over a field $\mathcal{F}$. Let $\mathcal{S}$ be a subset of $\mathcal{V}$. Let the vector addition in $\mathcal{S}$ be defined in the same way as in $\mathcal{V}$. If $\mathcal{S}$ is also vector space over the same field $\mathcal{F}$, we say that $\mathcal{S}$ is* **subspace** *of $\mathcal{V}$.*

Examples Of Subspaces

1. Every vector space $\mathcal{V}$ is subspace of itself.

2. The subset having only the null vector, 0, is a subspace of every vector space.

3. Let $\mathcal{V}_1$ be the vector space of complex numbers over the field of real numbers. Let $\mathcal{V}_2$ be the vector space of all real numbers with $\mathbb{R}$ as the field of scalars. The $\mathcal{V}_2$ is a subspace of $\mathcal{V}_1$.

4. The set $C^{(1)}$ of functions with continuous first derivative is a subspace of the vector space of all continuous functions with the same field of scalars.

5. Let $C^{(0)}$[a,b] be the set of all continuous complex valued functions on the interval $[a,b]$. This set is a vector space and we have

   (a) the subset consisting of of all functions which vanish at a given point x_0 is a subspace.

   (b) the subset of $C^{(0)}$ consisting of all functions having value 1/2 at a point x_0 is <u>not a subspace</u>.

   (c) The set of all solutions of a linear differential equation

   $$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + y(x) = 0$$

   is a vector space.

6. Consider the set of all vectors in three dimensions, $\mathbb{R}^3$ which is real vector space. The subset S_1 of all vectors which are multiples of a fixed vector $\vec{A}$ and the subset S_2 of all vectors in a given fixed plane passing through the origin, and are two examples of subspaces of $\mathbb{R}^3$.

It is easy to see that intersection of two subspaces of a vector space is again a subspace.

Lesson-3 Linear Independence and Basis

Lesson Overview

Syllabus Linear independence, basis, dimension

Lesson Objectives You will learn definition of linear independence, basis and dimension.

Prerequisite Definition of Vector spaces

§3.1 Linear independence

Definition 6 *A set of vectors $\mathcal{S} = \{f_1, f_2, ...f_n\}$ is called* **linearly dependent set** *if $\exists$ a set of scalars $\alpha_1, \alpha_2, \dots$ such that not all α's are zero and*

$$\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0$$

Definition 7 *A set of vectors $\mathcal{S} = \{f_1, f_2, ...f_n\}$ is called* **linearly independent set** *if it is not a linearly dependent set. This means that a set X is linearly independent if*

$$\alpha_1 f_2 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0$$

implies $\alpha_1 = \alpha_2 = \cdots = 0$.

Definition 8 *Let $\{f_1, f_2, ..., f_m\}$ be a finite set of vectors in vector space V. Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be a set of scalars and $f \in \mathcal{V}$ be such that*

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_m f_m$$

Then we say that f is **linear combination** *of the vectors $f_1, f_2, \dots, f_m$.*

Properties of linearly independent set

1. If $f \in \mathcal{V}$ is a linear combination of $\{f_1, f_2, \dots\}$, then the scalars α_i in

$$f = \sum \alpha_i f_i$$

   are uniquely determined if and only if $\{f_1, f_2, \dots\}$ is a independent set.

2. If $\{f_i\}$ is a linearly independent set, a necessary and sufficient condition that $f \in \mathcal{V}$ be a linear combination of $\{f_i\}$ is that the set $\{f, f_i\}$ be linearly dependent.

3. Every set of vectors containing a linearly dependent set is also linearly dependent.

Definition 1 *A vector space is called* **finite dimensional** *if $\exists$ an integer N such that every set containing more than N elements is a linearly dependent set.*

§3.2 Basis and dimension

Definition 9 *A set of vectors $\mathcal{X}$ is called a* **basis** *in a vector space $\mathcal{V}$ if the following two properties are satisfied.*

- the set $\mathcal{X}$ is a linearly independent set, and
- every vector $f \in \mathcal{V}$ is a linear combination of vectors in $\mathcal{X}$, i.e.,

$$f = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$$

where $x_k \in \mathcal{X}$ for all $k = 1, 2, \ldots, n$.

Examples of Basis

1. Vectors $\{\vec{i}, \vec{j}, \vec{k}\}$ form a basi s for the set of all vectors in three dimension.
2. Any three vectors
3. Any three vectors which are not coplanar form a basis in the space of vectors in three dimension.
4. $\{1, x, x^2, ..., x^N\}$ is a basis in the space of all polynomials of degree N.
5. The set $\bigcup_n \{\cos nx, \sin nx\}$, where $n = 1, 2, 3, ...$, is a basis in space of all periodic functions on $[-\pi, \pi]$ with period 2π.
6. The vectors $\mathcal{E} = e_1, e_2, ..., e_N$ where

$$e_1 = (1, 0, 0, ..., 0) \ ; e_2 = (0, 1, 0, ...0) \ ; .... \ e_N = (0, 0, 0, ..., 1)$$

   form a basis in the vector space $\mathbb{C}^N$. This basis will be called the canonical basis or the standard basis.

7. The vectors $\mathcal{E} = \{e_1, e_2, ..., e_N\}$ also form a basis in R^N and in $\mathbb{Q}^N$.

Theorem 1 (Number of Elements in a Basis) *The number of elements in any one basis is equal to number of elements in every other basis.*

Definition 10 *For a finite dimensional space the number of elements in a basis is defined to be the* **dimension** *of the vector space.*

Summary of Properties of Bases

Given that a vector space $\mathcal{V}$ has dimension N we have the following properties.

1. *Every set containing $N+1$ or more vectors is a linearly dependent set.*
2. *A set of N vectors is a basis if and only if it is linearly independent.*
3. *A set of N vectors $\mathcal{X}$ is a basis iff every vector in $\mathcal{V}$ is linear combination of vectors in the set*
   $\mathcal{X}$.

Definition 11 *Let $S = \{f_1, f_2, \ldots, f_m\}$ be subset of a vector space. The* **linear span** *of S is the set of all vectors f such that f is linear combination of vectors $f_1, f_2, \ldots, f_m \in S$. Linear Span of $S = \{f | f = \sum_{k=1}^{m} \alpha_k f_k\}$ and $f_k \in V$ and $\alpha_k \in V\}$*

References

1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).

2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).

Lesson-4 Linear Functional, Dual Vector Space

Lesson Overview

Syllabus Linear functional; Dual vector space

Lesson Objectives You will learn definition of linear functionals and that the set of all linear functionals is a vector space called the dual vector space.

Prerequisite Definition of Vector spaces

---

§4.1 Linear functional

Definition 12 *A* **linear functional** *on a vector space is a mapping from the vector space to the field of scalars :*

$$\Psi : f \mapsto \Psi(f) \in \mathcal{F}$$

such that Ψ is linear:

$$\Psi(\alpha f + \beta g) = \alpha\Psi(f) + \beta\Psi(g)$$

Examples of functionals

1. The functional which assigns $0 \in \mathcal{F}$ to every vector is a linear functional.

2. In $\mathcal{C}^n$ let $x = (\xi_1, \xi_2, ..., \xi_n)$ and $\Psi(x)$ defined by

$$\Psi(x) = \alpha_1\xi_1 + \alpha_2\xi_2 + \ldots + \alpha_n\xi_n$$

   is a linear functional where $\alpha_k \in \mathcal{C}$.

3. In the function space $\mathcal{L}^2[a, b]$ given $f\mathcal{L}^2[a, b]$ define a functional Ψ by

$$\Psi(f) = \int_a^b g^*(x)f(x)dx,$$

   for a fixed $g \in \mathcal{L}^2[a, b]$, then Ψ is linear functional.

4. For $f \in R^n$ the functional Φ_1 and Φ_2 defined below are not linear functionals. Let $f = (\alpha_1, \alpha_2, ..., \alpha_n)$ and

$$\Phi_1(f) = \sum_k |\alpha_k|; \qquad \Phi_2(f) = \sum_k |\alpha_k|^2$$

5. Given a vector x_0, we define a functional Ψ_0 by

$$\Psi_{x_0}(f) = \begin{cases} 1 & \text{if } f = x_0 \\ 0 & \text{if } f \neq x_0 \end{cases} \tag{1}$$

   It is easy to check that the functional $\Psi(x_0)$ is a linear functional.

6. In the vector space, $\mathbb{R}^3$, of all real vectors in 3 dimension a linear functional can be defined as follows.

   Choose a vector $\vec{X} \in \mathbb{R}^3$ and define a functional Ψ_X by

$$\Psi_X(\vec{A}) = \vec{X} \cdot \vec{A}, \qquad \forall \vec{A} \in \mathbb{R}^3$$

Equality of two functionals: Two linear functionals Ψ and Φ on a vector space $\mathcal{V}$ are said to be equal if $\Psi(f) = \Phi(f), \quad \forall f \in \mathcal{V}$.

§4.2 Dual vector space

Dual Vector Space

We will now define twin operations involving linear functionals and scalars
(i) addition of two linear functionals and
(ii) multiplication of a linear functional by a scalars

Given two arbitrary linear functionals Ψ and Φ, their sum $\Psi + \Phi$ is defined by giving its action on an arbitrary vector $f \in \mathcal{V}$ as

$$(\Psi + \Phi)f = \Psi(f) + \Phi(f).$$

The sum of two linear functionals is again a linear functional. Given a scalar α and a linear functional Ψ, multiplication of linear functional by scalar α, $(\alpha\Psi)$ is defined by

$$(\alpha\Psi)(f) = \alpha\Psi(f)$$

and the product $\alpha\Psi$ is again a linear functional. Then we have the following result:

Theorem 2 *The set of all linear functional on a vector space $\mathcal{V}$ forms a vector space. This vector space is called* **vector space dual** *to $\mathcal{V}$ and is denoted by $\widetilde{\mathcal{V}}$.*

The proof is easy. We need to verify that sum of two linear functionals and multiplication of a linear functional by a scalar result in linear functionals. The proof is obvious. Still let us write it down.

Checking linearity of $\Psi_1 + \Psi_2$: Let Ψ_1, Ψ_2 be two linear functionals. Consider $\Phi = \alpha_1\Psi_1 + \alpha_2\Psi_2$, then for all $f \in \mathcal{V}$

$$\begin{aligned}(\alpha_1\Psi_1 + \alpha_2\Psi_2)f &= (\alpha_1\Psi_1)(f) + (\alpha_2\Psi_2)(f) \quad (2)\\ &= \alpha_1\Psi_1(f) + \alpha_2\Psi_2(f). \quad (3)\end{aligned}$$

It can be proved that the dimension of a vector space dual to $\mathcal{V}$ is equal to the dimension of the vector $\mathcal{V}$ space itself. Remember that every vector space has a basis. So we can ask for a basis for the dual vector space. A useful construction of a basis in the dual space starts with a basis $\mathcal{B}$ in the vector space, and the basis obtained will be called basis dual to $\mathcal{B}$.

Definition 13 Dual Basis: *Let $\mathcal{B} = x_1, x_2, \ldots, x_N \subset \mathcal{V}$ be a basis. Let linear functionals $\Psi_1, \Psi_2, \ldots$ be defined, (as in Eq.(1)), by*

$$\Psi_1(f) = \begin{cases} 1 & \text{if } f = x_1 \\ 0 & \text{if } f \neq x_1 \end{cases} \quad (4)$$

$$\Psi_2(f) = \begin{cases} 1 & \text{if } f = x_2 \\ 0 & \text{if } f \neq x_2 \end{cases} \quad (5)$$

etc. In general, we have

$$\Psi_k(f) = \begin{cases} 1 & \text{if } f = x_k \\ 0 & \text{if } f \neq x_k \end{cases} \quad (6)$$

where $k = 1, 2, \cdots, N$. Then you can check that Ψ_k is a basis in the dual vector space. It is called basis dual to the chosen basis $\mathcal{B}$.

The definition of dual basis can be summarized as in the table given below.

$$
\begin{array}{c|cccccc}
 & x_1 & x_2 & ... & x_k & ... & x_N \\
\hline
\Psi_1 & 1 & 0 & ... & 0 & ... & 0 \\
\Psi_2 & 0 & 1 & ... & 0 & ... & 0 \\
... & 0 & 0 & ... & 0 & ... & 0 \\
\Psi_k & 0 & 0 & ... & 1 & ... & 0 \\
... & 0 & 0 & ... & 0 & ... & 0 \\
\Psi_N & 0 & 0 & ... & 0 & ... & 1
\end{array}
\tag{7}
$$

Prove it now: Show that the dual basis as defined as above is in fact a basis by proving the two properties, linear independence and spanning the whole space, that a basis must have.

)‖(**Short Examples 1** (Dual Basis) *In the vector space* $\mathbb{R}^3$*, given a basis* $\{\vec{A}, \vec{B}, \vec{C}\}$ *the set of vectors*

$$\vec{a} = \frac{\vec{b} \times \vec{c}}{|\vec{b} \times \vec{c}|}, \qquad \vec{b} = \frac{\vec{c} \times \vec{a}}{|\vec{c} \times \vec{a}|}, \qquad \vec{c} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}. \tag{8}$$

define linear functionals in the sense of <Example 5> on page 12.

Remark: The dual of dual of vector space $\mathcal{V}$ is the vector space $\mathcal{V}$ itself.

References

1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).

2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).

Lesson-5 New Vector Spaces from Old

Lesson Overview

Syllabus Sum of subspaces; Quotient spaces; Tensor product of vector spaces.

Lesson Objectives You will learn construction of new vector spaces from a given set of vector spaces.

Prerequisite Definition of Vector spaces; Equivalence relation

---

References

1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).
2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).

---

§5.1 Sum of subspaces

Definition 14 *If $\mathcal{M}, \mathcal{N}$ are two subspaces of a vector space $\mathcal{V}$, the* **sum** *of the subspaces $\mathcal{M}, \mathcal{N}$ is defined as the the linear space spanned by $\mathcal{M} \cup \mathcal{N}$. This space will be denoted by $\mathcal{M} + \mathcal{N}$.*

The subspace $\mathcal{M} + \mathcal{N}$ is same as the set of all vectors of the form $f + g$ where $f \in \mathcal{M}$ and $g \in \mathcal{N}$. If $f \in \mathcal{M} + \mathcal{N}$, then f will be of the form $x + y$ where $x \in \mathcal{M}$ and $y \in \mathcal{N}$. However, in general, x and y will not be determined uniquely by f. The decomposition will be unique if and only if the two subspace are disjoint, i.e., $\mathcal{M} \cap \mathcal{N} = \{0\}$. In such a case we shall use the notation $\mathcal{M} \oplus \mathcal{N}$.

)|||(**Short Examples 2** *Given a vector space $\mathcal{V}$ and a subsace $\mathcal{M}$, in general, there are several subspaces $\mathcal{N}$ such that $\mathcal{M} \oplus \mathcal{N} = \mathcal{V}$ and the decomposition of $\mathcal{V}$ into a sum is not unique. As an example, let us take the vector space to be $\mathbb{R}^2$. Here subspaces are straight lines passing through the origin. A general element of $\mathbb{R}^2$ is (α, β). Let us take the subspace $\mathcal{M}$ to be the $x-$ axis $(\alpha, 0)$, the the subspace $\mathcal{N}$ can be any other line passing through the origin. Suppose we take $\mathcal{N}$ to be a line with slope μ. A general element $x = (\alpha, \beta) \in \mathcal{V}$ can be trivially written as $x = f + g$ where*

$$f = (\alpha - \gamma, 0) \qquad \text{and} \qquad g = (\gamma, \beta).$$

For any given $\mu \neq 0$, the choice $\gamma = \beta/\mu$ gives g in the subspace $\mathcal{N}$.

Definition 15 *Let $\mathcal{U}$ and $\mathcal{V}$ are two vector spaces over the same vector field Let $\mathcal{W}$, be the set of all ordered pairs $\langle f, g\rangle$ with $f \in \mathcal{U}$ and $g \in \mathcal{V}$. The set $\mathcal{W}$ becomes a vector space if the vector addition in $\mathcal{W}$ is defined by*

$$\alpha_1\langle f_1, g_1\rangle + \alpha_2\langle f_2, g_2\rangle = \langle \alpha_1 f_1 + \alpha_2 f_2, \alpha_1 g_1 + \alpha_2 g_2\rangle$$

The vector space $\mathcal{W}$ so obtained is called direct sum of vector spaces $\mathcal{U}$ and $\mathcal{V}$ and we write $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$.

)|(***Short Examples 3*** *Let us take direct sum of vector spaces $\mathcal{U} = \mathbb{R}$ and $\mathcal{V} = \mathbb{R}^2$, both over field of real numbers $\mathbb{R}$. General element of the two spaces are $x \in \mathcal{U}$ and $(y, z) \in \mathcal{V}$. The dirct sum is the set of all elements (x, y, z) and is the three dimensional real space $\mathbb{R}^3$.*

The dimension of the direct sum $\mathcal{U} \oplus \mathcal{V}$ is $M + N$ where M and N are the dimensions of the two spaces. In fact, if $\mathcal{X} = \{x_1, x_2, \ldots, x_M\}$ is a basis in $\mathcal{U}$ and $\mathcal{Y} = \{y_1, y_2, \ldots, y_N\}$ is a basis in $\mathcal{V}$ then a basis in the direct sum space is

$$\{\langle x_1, 0\rangle, \langle x_2, 0\rangle, \ldots, \langle 0, y_1\rangle), \langle 0, y_2\rangle, \ldots \langle 0, y_N\rangle\}.$$

This can be proved easily and it shows that the dimension of the direct sum space is the sum of individual dimensions.

Let $\mathcal{V}$ be vector space and $\mathcal{M}$ be its subspace over field $\mathcal{F}$. We shall define a new vector space $\mathcal{N}$ such that $\mathcal{V}$ can be written as direct sum of $\mathcal{M}$ and $\mathcal{N}$. This construction is natural and a very general one.

§5.2 Quotient space

Given a vector space $\mathcal{V}$ and a subspace $\mathcal{M}$ we now introduce a relation on vectors in $\mathcal{V}$ by

$$f \sim g \qquad \text{if } f - g \in \mathcal{M}$$

The relation $\sim$ is an equivalence relation. To show this consider

- $\forall f \in \mathcal{V}$ we have $\quad f - f = 0 \in \mathcal{M}(\because \mathcal{M}\text{is a subspace}) \qquad \therefore f \sim f$
- If $f \sim g$ we have $\quad f - g \in \mathcal{M} \quad \Rightarrow g - f \in \mathcal{M} \quad \Rightarrow g \sim f$
- Let $f \sim g$ and $g \sim h$, then $(f - g) \in \mathcal{M}$ and $(g - h) \in \mathcal{M}$
  It then follows that

$$(f - g) - (g - h) \in \mathcal{M} \Longrightarrow (f - h) \in \mathcal{M} \Longrightarrow f \sim h$$

It, therefore, follows that $\sim$ is an equivalence relation. Every equivalence relation on a set *partitions* the set into mutully disjoint classes called equivalence classes. Let $[f]$ denote the equivalence class of the element $f \in \mathcal{V}$. Let the set of all these equivalence classes be denoted by $\mathcal{Q}$.

It, therefore, follows that $\sim$ is an equivalence relation. Every equivalence relation on a set *partitions* the set into mutually disjoint classes called equivalence classes. Let $[f]$ denote the equivalence class of the element $f \in \mathcal{V}$.

Let the set of all these equivalence classes We now define an operation of addition on the set $\mathcal{Q}$ of equivalence classes. Let The vector sum, $[f]+[g]$, of two classes is defined by selecting vectors $x \in [f]$ and $y \in [g]$ and taking the equivalence class [f]+[g] to be the class containg the vector $x+y$. The result is in fact independent of the chioce of elements in the two classes $[f]$ and $[g]$. Hence we can write

$$[f]+[g]=[f+g]$$

Similarly, if $\alpha \in \mathbb{F}$ and $[f]$ is any equivalence class, the the product $\alpha[f]$ is defined by selecting an element $x \in [f]$ and taking the equivalence class of αx, *i.e.*,

$$\alpha[f]=[\alpha x]$$

where x is any element of [f]. Again the result of the scalar multiplication operation as defined above is independent of the choice of element $x \in [f]$.

Definition 16 *With the operation of adding classes as above, the space* $\mathcal{Q}$ *becomes a vector space called* **quotient space** *and will be denoted by* $\mathcal{Q} = \mathcal{V}/\mathcal{M}$.

We leave it for the reader to verify that $\mathcal{Q}$ is indeed a vector space over the same field $\mathcal{F}$ and that the dimension of the quotient space is $N-M$ where $N=\dim(\mathcal{V})$ and $M=\dim(\mathcal{M})$.

)|||(**Short Examples 4** *Let* $\mathcal{V}$ *be two dimensional plane* $\mathbb{R}^2$ *and* $\mathcal{M}$ *be the x-axis,* $\mathbb{R}$.

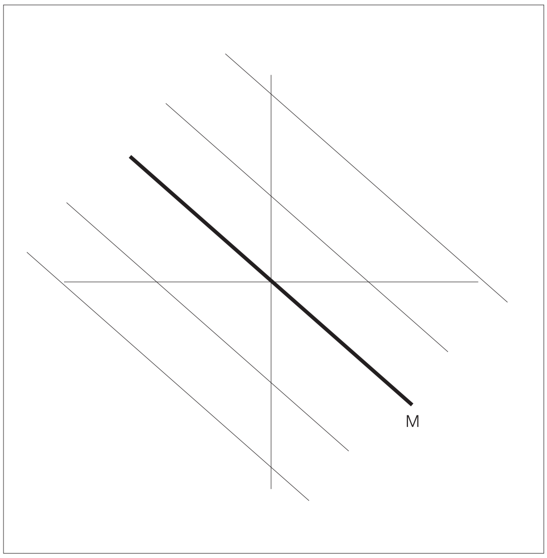
Then

$$\mathcal{V}=\{(x,y)|x,y\in\mathbb{R}\}\text{and}\mathcal{M}=\{(x,0)|x\in\mathbb{R}\}$$

The equivalence relation,$f \sim g$,$f \sim g$, *between two vectors* $f=(\alpha,\beta), g=(\alpha',\beta')$, *means* $\beta=\beta'$. *The equivalence class,* $[x]$, *of a vector* $x=(\xi,\eta)$ *consists of all vectors,* (ξ,η), *with varying* ξ *and a fixed* η. *Thus*

$$[x]=\{(\xi,\eta)|\xi\in\mathbb{R}\}$$

there being one class for each value of η. *It is obvious that the class* $[x]$ *is a straight line parallel to the x- axis and at a distance* η *from it.*



Quotient Space of Plane w.r.t. a line M passing through the Origin
The elements of the quotient space are lines parallel to M

Fig. ***1***

In general any line,M, through the origin is a subspace.The elements of the quotient space are the equivalence classes are which lines parallel to M (See Fig.1) . Each of these lines is completely specified by its distance from M. Thus the quotient space is isomorphic to the set of all real numbers. The vetor addition of the equivalence classes in the quotient space is just the addition of real numbers. Hence $\mathbb{R}^2/M$ *is the vector space* $\mathbb{R}$*.*

§5.3 Tensor product of vector spaces

We shall now introduce tensor product $\mathcal{V} \otimes \mathcal{U}$ *of two vector spaces* $\mathcal{U}$ *and* $\mathcal{V}$*. The correct way of defining the tensor product is to define it is through the space dual to the Cartesian product of the two vector spaces* $\mathcal{U}$ *and* $\mathcal{V}$*. We shall not describe this approach here but will be content with a 'working' definition only. For vectors* $x \in \mathcal{V}$ *and* $y \in \mathcal{V}$ *we formally introduce 'tensor product'* $z = x \otimes y$ *having the following properties.*

- $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$
- $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$
- $(\alpha x) \otimes y = \alpha(x \otimes y)$
- $x \otimes (\alpha y) = \alpha(x \otimes y)$

In general we have

$$\begin{aligned}(\alpha_1 x_1 + \alpha_2 x_2) \otimes (\beta_1 y_1 \otimes \beta_2 y_2) &= \alpha_1\beta_1(x_1 \otimes y_1) + \alpha_2\beta_1(x_2 \otimes y_1) \\ &\quad + \alpha_1\beta_2(x_1 \otimes y_2) + \alpha_2\beta_2(x_2 \otimes y_2) \quad (9)\end{aligned}$$

Definition 17 *Consider two vector spaces* $\mathcal{U}$ *and* $\mathcal{V}$ *of dimensions* M *and* N *respectively. Let* $\mathcal{X} = \{x_1, x_2, \cdots, x_N\}$ *and* $\mathcal{Y} = \{y_1, y_2, \cdots, y_M\}$ *be basis sets in the respective spaces. Let* z_{ij} *stand for a formal product, written as* $z_{ij} = x_i \otimes y_j$*. We regard the set* $\{z_{ij}\}$ *as a basis of a vector space* $\mathcal{U} \otimes \mathcal{V}$*. Note that not all sets of basis have vectors of tensor product type.*

)||(***Short Examples 5*** *As an example consider the spaces* $\mathcal{P}_N(x), \mathcal{P}_M(y)$ *of all polynomials in* x *of degree less than or equal to* N *and in* y *of degree less than or equal to* M*. The tensor product of two polynomials* $p(x) \in \mathcal{P}_N(x)$ *and* $q(y) \in \mathcal{P}_M(y)$ *be defined as*

$$p(x) \otimes q(y) = p(x)q(y)$$

This gives rise to the tensor product space $\mathcal{P}_N \otimes \mathcal{P}_M$ *whose elements are all polynomials in* x, y *of degree* N *in* x *and degree* M *in* y*. Give three different basis sets in this space.*

The definition of direct sum and tensor products can be extended to more than two spaces.