Bundled Lessons in Vector Spaces

Bundle -II Linear Operators

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1 Linear Operators on a Vector Space

Lesson Overview

Syllabus Linear operators; Sum product and commutator of two operators; Inverse of an operator; Properties of inverse of an operator.

Lesson Objectives You will learn construction of new vector spaces from a given set of vector spaces.

Prerequisite Definition of Vector spaces; Mapping of sets

References

1. Halmos P. R. *Finite Dimensional Vector Spaces* Springer Verlag, East West Edition (1974).
2. Fraleigh J. B. *A First Course in Abstract Algebra*, Pearson Education Limited, Essex, (2014).

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§1.1 Linear Operators-I

Basic definitions

Definition 1 *An* **operator***, T, on a vector space $\mathcal{V}$ is a mapping*

$$T : \mathcal{V} \to \mathcal{V}$$

from the vector space $\mathcal{V}$ into itself. In other words, to an arbitrary vector f from the vector space, an operator, T, assigns a unique vector, Tf, in the vector space $\mathcal{V}$.

$$T : f \mapsto Tf \in \mathcal{V}$$

Definition 2 *An operator, T on a vector space is a* **linear operator** *if it satisfies the property*

$$T(\alpha f + \beta g) = \alpha\ Tf + \beta\ Tg$$

$\forall \alpha, \beta \in \mathcal{F}$ *and* $\forall\ g \in \mathcal{V}$. *Equivalently an operator T is linear if*

$$T(f + g) = Tf + Tg \text{ and } T(\alpha f) = \alpha T f$$

It is, therefore, seen that for an operator T to be linear it is necessary that $Tf = 0$ if $f = 0$.

Definition 3 *Given two linear operators A and B we can define their* **sum** *, $A+B$, by means of the following rule for its action on an arbitrary vector.*

$$(A + B)f = Af + Bf$$

The sum of two linear operators is again a linear operator.

Definition 4 **Multiplication of a linear operator** *T by a scalar α is a linear operator defined by*

$$(\alpha T)f = \alpha(Tf)$$

Theorem 1 *With addition of linear operators and scalar multiplication defined as above, the set of all linear operators on a vector space $\mathcal{V}$ is again a vector space. If the dimension of the vector space $\mathcal{V}$ is n, the dimension of the vector space of all the operators on$\mathcal{V}$ is n^2.*

Product and commutator

Definition 5 **Product of two operators**, *A and B, is defined as in the case of mappings.*

$$(AB)f = A(Bf)$$

When A and B are linear operators, the product AB is also a linear operator.

Definition 6 *The* **commutator** *of two operators is defined to be*

$$[\,A, B\,] = AB - BA$$

Definition 7 *The* **anticommutator** *is defined by*

$$[\,A, B\,]_+ = AB + BA$$

Properties of commutator

The commutator satisfies the following properties.

$$\begin{aligned}
[\,A, B\,] &= -[\,B, A\,] && (1)\\
[\,\alpha_1 A_1 + \alpha_2 A_2, B\,] &= \alpha_1[\,A_1, B\,] + \alpha_2[\,A_2, B\,] && (2)\\
[\,A, \beta_1 B_1 + \beta_2 B_2\,] &= \beta_1[\,A, B_1\,] + \beta_2[\,A, B_2\,] && (3)\\
[\,A, BC\,] &= B[\,A, C\,] + [\,A, B\,]C && (4)\\
[\,AB, C\,] &= A[\,B, C\,] + [\,A, C\,]B && (5)
\end{aligned}$$

$$[\,A, [\,B, C\,]\,] + [\,B, [\,C, A\,]\,] + [\,C, [\,A, B\,]\,] = 0 \qquad (6)$$

The last relation is known as the Jacobi identity.

Definition 8 *With sum, product and multiplication by a scalar defined for operators, the following expression defines* **polynomial** *in a linear operator A.*

$$p(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + +\alpha_n A^n$$

The operator $p(A)$ is again a linear operator.

§1.2 Inverse of an operator-Basics

Definition 9 *Let T be an operator on a vector space. We say T is* **one to one** *if action of T on two* distinct vectors *gives* distinct answers.

$$x_1 \neq x_2 \implies Tx_1 \neq Tx_2$$

This is equivalent to the condition

$$Tx_1 = Tx_2 \Longrightarrow x_1 = x_2$$

Definition 10 *An operator T on a vector space is called* **onto** *if $\forall\ y \in \mathcal{V}$ we can find at least one $x \in \mathcal{V}$ such that $Tx = y$. This x may, in general, not be unique.*

Definition 11 *An operator is called* **invertible** *if it is both one to one and onto.*

Definition 12 *Let T be an operator which is both one to one and onto. We define* **inverse** *of T by giving its action on an arbitrary vector $u \in \mathcal{V}$.*

Because T is onto, we can find a vector u such that $Tu = v$. Since T is one to one it follows that u satisfying $Tu = v$ is uniquely determined once the vector v is specified. We define **inverse** of T, to be denoted by T^{-1}, by the equation

$$T^{-1}v = u$$

This definition coincides with the defintion of the inverse for a mapping. The inverse satisfies

$$(AB)^{-1} = B^{-1}A^{-1},$$

$$(\alpha A)^{-1} = (1/\alpha)A^{-1}, \alpha \neq 0.$$

Definition 13 *Let T be a linear operator on a vector space $\mathcal{V}$. The* **range** *$\mathcal{R}(T)$ is the set of vectors obtained by applying T on all vectors $f \in \mathcal{V}$.*

$$\mathcal{R}(T) = \{g | g = Tf, \in \mathcal{V}\}$$

Definition 14 *Also the* **null space** *of an operator, $\mathcal{N}(T)$, is the set of all those vectors x for which $Tx = 0$.*

$$\mathcal{N}(\ T) = \{x | x \in \mathcal{V} \text{ and } Tx = 0\}$$

Both $\mathcal{R}(T)$ and $\mathcal{N}(T)$ are subspaces of the vector space $\mathcal{V}$. (Proof ?!)

Definition 15 *The dimension of $\mathcal{R}(T)$ for an operator is called the rank of the operator T. Obviously rank $(T) \leq \dim \mathcal{V}$.*

Clearly rank of an operator is the maximum number of linearly independent vectors that can be selected from Tf when varies over the entire vector space $\mathcal{V}$.

§1.3 Inverse of an operator-Properties

An operator on a vector space is invertible if it is one to one and onto operator. For *linear operators in finite dimensional vector spaces* the property of being one to one is equivalent to the onto property. Thus a linear operator on a finite dimensional space is invertible if anyone of the of the following statements holds.

(a) the operator is one to one

(b) the operator is onto.

Now we will prove a set of theorems which provide necessary and sufficient conditions so that a linear operator in a finite dimensional vector space may be invertible.

Theorem 2 *Let T, S, R be arbitrary (linearity is not demanded) operators on a vector space such that*

$$TS = RT = I \tag{7}$$

where I is the identity operator. Then T is invertible and

$$R = S = T^{-1}$$

Proof:

We begin with noting the linearity is not demanded as a condition on the operators. Let R and S exist such that Eq.(7) is satisfied. Then

$$T(Sx) = (TS)x = x, \forall x \in \mathcal{V}.$$

Thus T is onto because given any vector $x \in V$ there exists a vector ($y = Sx$) such that $Ty = x$.

Next using $RT = I$ we will show that T is one to one. To show that T is one to one, we must prove that $Tx_1 = Tx_2 \iff x_1 = x_2$. Let $Tx_1 = Tx_2$ apply R on both sides. This gives $R(Tx_1) = R(Tx_2)$, or, $(RT)x_1 = (RT)x_2$ using the given property $RT = I$ we get the desired result that $x_1 = x_2$. Thus $Tx_1 = Tx_2 \iff x_1 = x_2$ .Therefore T is one to one. Thus T is invertible because T is both one to one and onto.

Conversely if T^{-1} exists , the given relations are satisfied for $R = S = T^{-1}$. There (II.12) is necessary and sufficient for an operator to have an inverse. Note that when the operator T is linear, each *one* of the two conditions (a) $TS = I$ (b) $RT = I$ is *separately* sufficient for T to have an inverse.

Theorem 3 *Let T be a linear operator on a finite dimensional vector space $\mathcal{V}$ $(\dim V = N)$. Then the following statements are equivalent.*

(2.1) $Tx = 0$ implies $x = 0$.

(2.2) T is one to one.

(2.3) If $\mathcal{X} = \{x_1, x_2, \ldots\}$ is a linearly independent set then $T\mathcal{X} \equiv \{Tx_1, Tx_2, \ldots\}$ is also linearly independent set.

(2.4) If $\mathcal{X} = \{x_1, x_2, \ldots, x_N\}$ is a basis then $T\mathcal{X} \equiv \{Tx_1, Tx_2, \ldots, Tx_N\}$ is also a basis.

(2.5) T is an onto operator.

(2.6) Let $\mathcal{X} = \{x_1, x_2, \ldots,\}$ and $\mathcal{Y} = \{y_1, y_2, \ldots,\}$ be sets of vectors such that $y_j = Tx_j$. If $\mathcal{Y}$ is a linearly independent set of vectors, then $\mathcal{X}$ is also linearly independent.

(2.7) If $\mathcal{X}$ and $\mathcal{Y}$ are as in (2.6) above and if $\mathcal{Y}$ is a basis, then $\mathcal{X}$ is also a basis.

Proof:

We shall prove that
(2.1) $\Longrightarrow$ (2.2) $\Longrightarrow$ (2.3) $\Longrightarrow$ (2.4) $\Longrightarrow$ (2.5) $\Longrightarrow$ (2.7) $\Longrightarrow$ (2.1)

Proof of (2.1) $\Longrightarrow$ (2.2) We are given (2.1) :$Tx = 0 \Longrightarrow x = 0$. Consider $Tx_1 = Tx_2$; Then $Tx_1 - Tx_2 = 0$ Using linearity we get $T(x_1 - x_2) = 0$. Using (2.1) we get $x_1 = x_2$. Thus $Tx_1 = Tx_2 \rightarrow$ This means that T is one to one.

Proof of (2.2) $\Longrightarrow$ (2.3) To prove $\{Tx_1, Tx_2, ...\}$is linearly independent consider

$$\alpha_1(Tx_1) + \alpha_2(Tx_2) + ... = 0 = T0$$

$$T(\alpha_1 x_1 + \alpha_2 x_2 + ...) = T0$$

Using (2.2), T is one to one gives $\alpha_1 x_2 + \alpha_2 x_2 + ... = 0$ It is given that $x_1, x_2, ...$ is linearly independent, hence we get $\alpha_1 = \alpha_2 = ... = 0$ This proves that $\{Tx_1, Tx_2, ...\}$ is linearly indepndent.

Proof of (2.3) $\Longrightarrow$ (2.4) Let $B = \{e_1, e_2, ..., e_N\}$ be a set of basis vectors. $\therefore \{e_1, e_2, ..., e_N\}$ is linearly independent. Hence using (2.3) we see that $\{Te_1, Te_2, ..., Te_N\}$ is also linearly independent set. The number of elements, N, in this set is equal to the dimension of the vector space, hence the set is also a basis set.

Proof of (2.4) $\Longrightarrow$ (2.5) To prove (2.5),*i.e.*, T is onto we must show that $\forall$ vectors $y \in \mathcal{V}$, we can find a vector x such that $Tx = y$. Let$\{e_1, e_2, ..., e_N\}$ be a basis set then (2.4) gives that $\{Te_1, Te_2, ..., Te_N\}$ is also a basis. $\therefore$ given an arbitrary vector $y \in \mathcal{V}$, we can expand y in terms of the vectors $\{Te_1, Te_2, ..., Te_N\}$:

$$y = \alpha_1 Te_1 + \alpha_2 Te_2 + ... + \alpha_N Te_N$$

or , using linearity we have

$$y = T(\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_N e_N)$$

Therefore, we have shown that y can be written as Tx, where x is given by

$$x = (\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_N e_N)$$

Proof of (2.5) $\Longrightarrow$ (2.6) Given T is onto means that for every $y \in \mathcal{V}$ we can find at least one vector $x \in \mathcal{V}$ such that $y = Tx$. Hence, starting from $\{y_1, y_2, ...\}$ we can form the set $\{x_1, x_2, ...\}$ such that $Tx_k = y_k$. Assume, as given in (2.6), that $\{y_1, y_2, ...\}$ is LI . To prove that $\{x_1, x_2, ...\}$ is LI consider

$$\alpha_1 x_1 + \alpha_2 x_2 + ... = 0$$

Applying T on the above equation we get

$$T(\alpha_1 x_1 + \alpha_2 x_2 + ...) = 0$$

or, using the linearity

$$\alpha_1 T x_1 + \alpha_2 T x_2 + ... = 0$$

or,

$$\alpha_1 y_1 + \alpha_2 y_2 + ... = 0$$

As $\{y_1, y_2, ...\}$ is given to be linearly independent the above relation can be satisfied only when

$$\alpha_1 = \alpha_2 = ... = 0$$

This proves that the set $\{x_1, x_2, ...\}$ is LI .

Proof of (2.6) $\Longrightarrow$ (2.7) $\mathcal{Y} = \{y_1, y_2, ..., y_N\}$ is a basis set and is, therefore, LI. (2.6) gives that the set $\mathcal{X} = \{x_1, x_2, \ldots, x_N\}$ is also LI. Linear independence of the set $\mathcal{X}$ along with the fact that the number of elements in $\mathcal{X}$ is equal to the dimension of the vector space $\mathcal{V}$ proves that the set $\mathcal{X}$ is a basis set.

Proof of (2.7) $\Longrightarrow$ (2.1) Let $\mathcal{Y} = \{y_1, y_2, ...y_N\}$ be a basis and let the set $\mathcal{X} = \{x_1, x_2, ..., x_N\}$ be as in the statement (2.7) then $\mathcal{X}$ is a basis.

Assume x is such that $Tx = 0$, we have to show that $x = 0$. Expand x in terms of vectors in the basis set $\mathcal{X}$:

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_N x_N$$

Applying T on the above equation, and using $Tx = 0$, we get

$$T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_N x_N) = 0$$

$$\alpha_1 T x_1 + \alpha_2 T x_2 + \ldots + \alpha_N T x_N = 0$$

$$\alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_N y_N = 0$$

The last equation above can hold only when

$$\alpha_1 = \alpha_2 = \ldots = \alpha_N = 0$$

This gives the desired result $x = 0$ proving (2.1).

We will now use the above statements to formulate the condition of existence of in inverse of an operator in terms of dimension of the vector space, range space of T, etc.

Theorem 4 *Let t be an invertible linear operator on a vector space $\mathcal{V}$ then the following five statements are equivalent to the existence of inverse of the operator T.*

3.1 The range space of T is entire vector space, i.e. $T\mathcal{V} = \mathcal{V}$.

(3.2) The null space of T is equal to $\{0\}$.

(3.3) $dim(T\mathcal{V}) = dim\mathcal{V}$

(3.4) Rank T = dim $\mathcal{V}$

(3.5) dim $\mathcal{N}(T) = 0$

Proof:

1. The property that the range space of T is entire space is equivalent to T being onto.

2. The property that the null space $\mathcal{N}(T) = \{0\}$, is just a restatement of $Tx = 0 \Rightarrow x = 0$.

3. Let $\mathcal{X} = \{x_1, x_2, ..., x_n\}$ be a basis set for the vector space $\mathcal{V}$. Now the following implications hold.
   T has inverse $\Longleftrightarrow$ T is one to one $\Longleftrightarrow$ $T\mathcal{X} = \{Tx_1, Tx_2, ..., Tx_n\}$ is a basis set
   $\Longleftrightarrow$linear span of $T\mathcal{X} = \mathcal{V} \Longleftrightarrow T\mathcal{V} = \mathcal{V}$ The last implication follows from the fact that $\mathcal{X} \subsetneqq \mathcal{V}$ implies that $T\mathcal{X} \subsetneqq \mathcal{V}$. This together with $T\mathcal{X} = \mathcal{V}$ gives $T\mathcal{V} = \mathcal{V}$.

4. For proving invertibility being equivalent to (3.4), recall that rank$T$ = dim$(T\mathcal{V}$. Thus (3.4) is true if and only if (3.1) is true.

5. The property (3.4) gives that a linear operator T is invertible if and only if rank T = dim$\mathcal{V}$. Now dim$\mathcal{N}(T)$ + rank(T) = dim$\mathcal{V}$. This proves that T is invertible if and only if dim $\mathcal{N}(T) = 0$. Thus existence of of inverse is equivalent to (3.5).

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2 Matrix Representation in a Basis