Chennai Mathematical Institute

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Quantum Field Theory $\boxed{\text{Class Room Example}}$ Feb 3, 2020

Problem in this set concerns computation of differential cross section for Coulomb scattering using second quantized Schrödinger field theory in the first order of perturbation theory in the interaction picture.
To help you formulate the problem, I will give the steps in this computation. The details can then be filled in.

I divide the steps into four parts for purpose of this tutorial. My list of steps as given here is long and detailed. Once you have understood the computation formulate your own steps and rules.

Step-I <u>Calculating the matrix element</u>

[1] The Lagrangian density for Schrodinger field is

$$\mathcal{L} = i\hbar\psi^*(\vec{r},t)\frac{\partial}{\partial t}\psi(\vec{r},t) + \frac{\hbar^2}{2m}\Big(\nabla\psi^*(\vec{r},t)\Big)\Big(\nabla\psi(\vec{r},t)\Big) - \psi^*(\vec{r},t)V(\vec{r})\psi(\vec{r},t) \tag{1}$$

where $V(\vec{r})$ external potential potential seen by the Schrodinger field. You will get the same result that is obtained by using the time independent approach and also by taking time dependent route via Fermi Golden rule in Schrodinger Wave mechanics. You potential to be Yukwa potential $V(r) = V_0\frac{e^{-\mu r}}{r}$ and compute the Rutherford scattering of alpha particles from a nucleus of charge Z by taking $V_0 = ZZ'$ and taking limit $\mu \to 0$.

[2] In interaction picture

$$H_I' = \exp(iH_0t/\hbar)H'\exp(-iH_0t/\hbar)$$

where H_0 is free Hamiltonian and interaction Hamiltonian is

$$H' = \int d^3x\psi^*(\vec{r},t)V(\vec{r})\psi(\vec{r},t)$$

[3] In the second quantisation scheme in interaction picture, expand the field in terms of plane waves and treat the expansion coefficients $a(\vec{k})$ as

$$\begin{aligned}\psi(\vec{r},t) &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} a(\vec{k})e^{i\vec{k}\cdot\vec{r}}\,d^3k &(2)\\ \psi^*(\vec{r},t) &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} a^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{r}}\,d^3k. &(3)\end{aligned}$$

The ETCR for the operators $a(\vec{k})$ and $a^\dagger(\vec{k})$ are given

$$\big[a(k), a^\dagger(q)\big] = \delta(\vec{k}-\vec{q}) \tag{4}$$

[4] Next compute the transition amplitude for the scattering by computing the matrix elements of the first order term in the perturbation series of the time evolution operator. Thus compute

$$m_{fi} = \frac{1}{i\hbar}\int_{-T/2}^{T/2} \langle f|\, H'_{\rm I}(t)\, |i\rangle\, dt \tag{5}$$

Here $|i\rangle$ and $|f\rangle$ denote the initial and final states of a single particle, with momenta $\vec{k}_i, \vec{k}_f$.

Use the fact that initial and final states are eigenstates of free Hamiltonian with energies $E_i = \hbar^2 k_i^2/2m, E_f = \hbar^2 k_f^2/2m$ respectively. Integrate over time and obtain the transition amplitude $m_{\rm fi}$ This should give a factor

$$\frac{\sin(\Delta E)T/2)}{\Delta ET/2}, \qquad \text{where } \Delta E = E_{\rm i} - E_{\rm f}.$$

Note that as $T \to \infty$, the above factor tends to Dirac delta $\delta(E_i - E_f)$ and squaring it would give a nonsensical answer.

This factor has to handled carefully as you might have done quantum mechanics while working with ttime depedent perturbation theory and Fermi Golden rule.

[5] Use

$$|i\rangle = a^\dagger(\vec{k}_i)|0\rangle, \qquad |f\rangle = a^\dagger(\vec{k}_f)|0\rangle,$$

and obtain the matrix element as function of momenta. Keep the integral over $V(r)$ as it is, don't try to do the space integral at this stage.

Step-II: Transition probability per unit time

Next square and compute the transition probability. It will have square of the factor $\frac{\sin(\Delta E)T/2)}{\Delta ET/2}$ as given above.

Find transition probability per unit time, differentiate $|m_{\rm fi}|^2$ w.r.t. time T and take limit $\Delta T \to \infty$. In this limit you should get a Dirac delta function $\delta(E_i - E_f)$.

Denote this transition probability per unit time as $w_{\rm fi}$

Sum over final states.

Step III: Sum over final states

[1] In a measurement of differential cross section all particles with for final momenta in the range $\vec{k}, \vec{k} + d\vec{k}$ are counted. Therefore we must integrate over all momenta in this range. Note that, in our notation

$$\sum_k (.) \to \iiint d^3k(.) \tag{6}$$

Change notation $k_f \to k$,

Relate the range d^3k to the solid angle as $k^2 dk\, d\omega$.

Use Dirac delta function to carry out the integral over k.

Remember delta function makes energies equal $E_f = E_i$, therefore $\therefore k_i = k_f = k$.

Step IV: Find cross sections and life times

[2] Imagine the scattering experiment is repeated by sending one particle N times, then
`Number of particles scattered into solid angle` $d\Omega$ `per sec`
`= N` $\times$ `transition probability per unit time to states`
`with final momentum in the solid angle` $d\Omega$.

Use the fact, that for differential cross section, the range of solid angle is small. So that integration over $d\Omega$ results in multiplying by $d\Omega$.

[3] Recall the definition of cross section.
`The number of particles scattered into solid angle` $d\Omega$
$= \texttt{Flux} \times d\Omega \times$ differential cross section,
and

`Flux =` $N\times$ `probability current`
where N is the total number of particles sent during the experiment. .

[4] Use (7) and (8) to find the differential cross section. Replace $V(r) = V_0 \exp(-\mu r)/r$. Compute the integral over spatial coordinates now.

[5] Take the limit $\mu \to 0$ and compare your answer with Rutherford formula for Coulomb scattering.

This division into four major steps will remain the same for all situations that we are going to deal with. Details will differ from case to case.