

# QFT Lecture Notes-II\*

## Part-II-Second Quantization (Course given at IIT Bhubaneswar)<sup>†</sup>

Space for your notes, comments, questions etc

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\*Some chapters, that were prerequisites, have been included in lecture notes for completeness.

## §1 Quantization of Schrodinger equation

### §1 Second Quantisation of Schrodinger equation

We will recall the main steps and features of second quantisation of non-relativistic Schrodinger equation. The non-relativistic Schrodinger equation for a single point particle moving in a potential  $V(\vec{x})$  is given by

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{x}, t) + V(\vec{x})\psi(\vec{x}, t). \quad (1)$$

with Lagrangian given by

$$L[\psi, \partial_k \psi, \psi] = \int \mathcal{L} d^3x \quad (2)$$

**Lagrangian formulation** The quantum state of a particle at any time is specified by the wave function at that time. In other words by the values of  $\psi(\vec{x})$  for different space time points  $\vec{x}$ . Therefore, we may regard the system as having infinite number of generalised coordinates  $\psi(\vec{x})$ , one for each point in space and seek to quantise the Schrodinger equation again. We shall follow the canonical quantisation scheme for this purpose.

The Lagrangian density giving rise to the Schrodinger equation as Euler Lagrange equation is

$$\mathcal{L} = i\hbar \frac{\partial \psi^*}{\partial t} \psi - \psi^* \left( \frac{\hbar^2}{2m} (\nabla \psi)^* (\nabla \psi) + V(\vec{x})\psi \right) \quad (3)$$

The Euler Lagrange equation is

$$\frac{\delta L}{\delta \psi(\vec{x}, t)} = 0 \quad (4)$$

The equation of motion, in terms of the Lagrangian density, is seen to be

$$\partial_0 \left( \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} \right) + \sum_k \partial_k \left( \frac{\partial \mathcal{L}}{\partial (\partial_k \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0. \quad (5)$$

It is easily checked that (5) leads to the Schrodinger equation when the Lagrangian density is given by Eq.(3) is used.

**Canonical momentum** For canonical quantisation, we compute the canonical momentum  $\pi(\vec{x}, t)$  conjugate to  $\psi(\vec{x}, t)$  using the definition

$$\pi(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}(\vec{x}, t)}. \quad (6)$$

and get

$$\pi(\vec{x}, t) = \psi^*(\vec{x}, t). \quad (7)$$

**Canonical quantisation** The act of quantisation consists in assuming that  $\psi(\vec{x}, t), \pi(\vec{x}, t)$  are operators obeying equal time commutation relations (ETCR)

$$[\psi(\vec{x}, t), \pi(\vec{y}, t)] = i\hbar\delta^{(3)}(\vec{x} - \vec{y}), \quad (8)$$

$$[\psi(\vec{x}, t), \psi(\vec{y}, t)] = 0, \quad [\pi(\vec{x}, t), \pi(\vec{y}, t)] = 0. \quad (9)$$

This assumption is very powerful and contains all information about the second quantised Schrodinger theory.

## §2 Building up the Hilbert space

The field operator  $\psi(\vec{x}, t)$  does not have any physical interpretation in the second quantized theory. The interpretation will become clear only after a certain amount of mathematical developments in the second quantised theory.

We would like to learn a little about the description of states, and about the Hilbert space of all possible states, of the second quantised theory.

To arrive at the Hilbert space of states we first expand the field operator as

$$\psi(\vec{x}, t) = \sum_n a_n(t) u_n(\vec{x}). \quad (10)$$

Here  $u_n(\vec{x})$  are the eigenfunction of the Hamiltonian operator

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x}). \quad (11)$$

in the first quantised theory. It has been assumed that all the eigenvalues are discrete and that the eigenfunctions  $u_n$  are normalized to unity.

Eq.(10) can be inverted and we will have

$$a_n(t) = \int d^x u_n(\vec{x}) \psi(\vec{x}, t). \quad (12)$$

Also its adjoint equation takes the form

$$a_n^\dagger(t) = \int d^x u_n(\vec{x}) \psi^\dagger(\vec{x}, t). \quad (13)$$

With  $\pi(\vec{x}, t)$  given by (7), the canonical commutation relation (8) takes the form

$$[\psi(\vec{x}, t), \psi^*(\vec{y}, t)] = i\hbar \delta^{(3)}(\vec{x} - \vec{y}). \quad (14)$$

This can be used to find the commutation relation satisfied by the operators  $a_n$  and  $a_n^\dagger$ . It is now straightforward to verify the following commutation relations.

$$[a_m, a_n^\dagger] = \delta_{mn}, \quad (15)$$

$$[a_m, a_n] = 0, \quad [a_m^\dagger, a_n^\dagger] = 0, \quad (16)$$

where  $m, n = 1, 2, \dots$ . Next we define the operators  $N_k = a_k^\dagger a_k$ . These are commuting hermitian operators

$$[N_k, N_\ell] = 0, \quad \text{for all } k, \ell. \quad (17)$$

It will be noticed that the operators  $a_k, a_k^\dagger, N_k$ , for each  $k$ , are like raising, lowering and number operators,  $a, a^\dagger, N = a^\dagger a$ , for the harmonic oscillator problem, and satisfy the same commutation relations. Therefore, we conclude that the eigenvalues of  $N_k$  are nonnegative integers and corresponding  $a_k, a_k^\dagger$  act like raising and lowering operators for the eigenvalues of  $N_k$ .

Thus we have a set of commuting hermitian operators  $\{N_k | k = 1, 2, \dots\}$  having nonnegative integers as eigenvalues. They will therefore have simultaneous eigenvectors which will form a complete orthonormal set. The Hilbert space of states will be linear span of this orthonormal set of simultaneous eigenvectors. These eigenvectors will be denoted as  $\{|\nu_1, \nu_2, \dots\rangle\}$ , so that

$$N_k |\nu_1, \nu_2, \dots\rangle = \nu_k |\nu_1, \nu_2, \dots\rangle. \quad (18)$$

$$\text{and } N |\nu_1, \nu_2, \dots\rangle = \sum_k \nu_k |\nu_1, \nu_2, \dots\rangle. \quad (19)$$

where  $N = \sum_k N_k$  is the sum of all number operators.

## Physical Interpretation

In order to learn about the physical interpretation we look at the observables of the second quantised theory. The Hamiltonian can be found using

$$H = \int d^x \left( \pi(\vec{x}, t) \frac{\partial \psi(\vec{x}, t)}{\partial t} - \mathcal{L} \right) \quad (20)$$

The Hamiltonian when expressed in terms of  $a_k, a_k^\dagger$  takes the form

$$H = \int d^3x \left( -\frac{\hbar^2}{2m} |\nabla \psi|^2 + \psi^*(\vec{x}, t) V(\vec{x}) \psi(\vec{x}, t) \right) \quad (21)$$

$$= \sum_k E_k N_k. \quad (22)$$

The lowest energy state corresponds to zero eigenvalue for operators  $N_k$  for each  $k = 1, 2, \dots$ . Assuming this state to be unique, and denoting it by  $|0\rangle$ , we have the property.

$$N_k |0\rangle = 0 \quad (23)$$

and therefore the energy of this state turns out to be zero. This state will be the ground state.

Next consider the state  $|0, 0, \dots, \nu_j, \dots\rangle$  which corresponds to eigenvalue  $\nu_j$  for a particular number operator  $N_j$  and zero for all other operators  $N_k, k \neq j$ . This state will result from repeated action of  $a_j^\dagger$  on the state  $|0\rangle$ :

$$|0, 0, \dots, \nu_j, \dots\rangle = C_j (a_j^\dagger)^{\nu_j} |0\rangle \quad (24)$$

where  $C_j$  is a constant, to be fixed by normalisation. This state has energy  $\nu_j E_j$  as is apparent from Eq.(22).

The state  $|0, 0, \dots, \nu_j, \dots\rangle$  has energy  $\nu_j E_j \equiv E$  and the action of  $a_j$  on this state results in a state with decreased energy  $E - E_j$ .

Also if  $|E\rangle$  is a state with total energy  $E$  then  $a_j^\dagger |E\rangle$  is a state with definite energy  $E + E_j$ .

This suggests the following interpretation.

- The eigenvalues of operators  $N_k$  gives the number of particles in level  $E_k$ .
- State  $|\nu_1, \nu_2, \dots\rangle$  corresponds to  $\nu_1, \nu_2, \dots, \nu_j, \dots$  particles in levels  $E_1, E_2, \dots, E_j, \dots$

- The state  $|0\rangle$  represents a state with zero energy and has no particle. This state is known as the vacuum state.
- The operators  $a_k^\dagger, a_k$ , respectively, create and annihilate a particle in  $k^{\text{th}}$  energy level  $E_k$ . level.
- The total number of particles,  $N = \sum_k N_k$ , is a constant of motion. In fact in absence of any other term in the Lagrangian each operator  $N_k$  is a constant of motion.

### §3 Fock Space

It has been noted in the above that the Hilbert space in the second quantised Schrodinger theory consists

- (i) a state with no particle  $|0\rangle$ ;
- (ii) states  $\{|\vec{q}\rangle\}$  of a single particle with all possible values of momenta  $\vec{q}$ ;
- (iii) states  $\{|\vec{q}\rangle, |\vec{p}\rangle\}$  of two particles having all possible values of momenta  $\vec{q}, \vec{p}$ ;
- (iv) and so on Mathematically, the Hilbert space consists of a direct sum

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$$

of several Hilbert spaces,  $\mathcal{H}_N$ , of states with  $N$  particles with  $N$  taking values  $0, 1, 2, \dots$ . The each Hilbert spaces  $\mathcal{H}_N$ , of states with with  $N$  particles, is *symmetric tensor product*

$$\mathcal{H}_N = \mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H}$$

of  $N$  copies of Hilbert space  $\mathcal{H}$  of single particle states.

The symmetric tensor product of Hilbert spaces,  $\mathcal{H}_n$ , of one particle states, is the same as that appears in the wave mechanics of identical bosons, described by symmetrised Schrodinger wave functions.

If it is desired that second quantised Schrodinger theory describe an assembly of identical fermions, the canonical commutation relations, in Eq.(8), must be replaced by anti-commutation relations.

$$[\psi(\vec{x}, t), \pi(\vec{y}, t)]_+ = i\hbar\delta^{(3)}(\vec{x} - \vec{y}), \quad (25)$$

$$[\psi(\vec{x}, t), \psi(\vec{y}, t)]_+ = 0, \quad [\pi(\vec{x}, t), \pi(\vec{y}, t)]_+ = 0. \quad (26)$$

In this case each of the Hilbert spaces  $\mathcal{H}_N$ ,  $N \geq 2$  will be antisymmetric tensor product of Hilbert space of one particle states.

The mathematical construction of the space of all states in the second quantized theory is known as *Fock space*. You can learn more about it from the references given at the end.

Remember that in the nonrelativistic description particles cannot be created or destroyed. The total number  $N$  of each type of particles in a system remains constant in time. The quantum mechanical description of such a system in the second quantised theory is completely equivalent to that in the Schrodinger wave mechanics.

In the Schrodinger wave mechanics, the (anti) symmetrisation of total wave function is an additional postulate. In the second quantised theory, it comes out automatically as a consequence of canonical (anti) commutation relations. In second quantised, relativistic theory, the spin statistics connection — use of commutators for bosons, and anticommutators for fermions — is a consequence of very general assumptions such as Lorentz invariance, positive definiteness of Hamiltonian and was first formulated and proved by Pauli and Luders in a paper in 1940 and has been later proved by others at different levels of mathematical rigour.

## §4 Relativistic Wave Mechanics

**Klein Gordon equation** Non relativistic Schrodinger equation does not explain all the fine details of H atom spectrum. The fine structure is explained by taking relativistic effects into account. Simplest such theory is based on the relativistic energy momentum relation  $E^2 = \vec{p}^2 c^2 + m^2 c^4$ . The usual prescription  $\vec{p} \rightarrow \hat{p} = -i\hbar\nabla$  and time evolution governed by the Hamiltonian

$$i\hbar \frac{\partial \phi(\vec{x}, t)}{\partial t} = \hat{H} \phi(\vec{x}, t) \quad (27)$$

gives us the equation

$$-\hbar^2 \frac{\partial^2 \phi(\vec{x}, t)}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \phi(\vec{x}, t) + m^2 c^4 \phi(\vec{x}, t) \quad (28)$$

This equation was originally written by Schrodinger himself. He abandoned its predictions for the fine structure did not agree with the experiments. Later he realized that in the non relativistic limit the H atom levels agreed with the Balmer formula. The equation (41) was not successful in describing relativistic quantum mechanics of a point particle.

What are the problems in using Klein Gordon equation as relativistic quantum mechanical wave equation? Where can we find a brief historical account?

Klein Gordon equation was revived as second quantized relativistic field equation for scalar particles.

We will briefly discuss the relativistic quantum mechanics of a single point particle. We will use natural units and set  $\hbar = c = 1$ . The Klein Gordon equation then takes the formula

$$(\square + m^2)\phi(\vec{x}, t) = 0 \quad (29)$$

where  $\square$  is given by

$$\square = \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (30)$$

and is known as de Alembertian operator. This equation describes relativistic quantum mechanics of a point particle.

**Plane wave solutions** The plane wave solutions are eigenfunctions of the momentum operators  $\vec{p} = -i\hbar\nabla$  and describe a relativistic free particle with definite momentum. The plane wave solutions of Klein Gordon equation are given by

$$f_q(\vec{x}, t) = N e^{i\vec{q}\cdot\vec{x} - i\omega_q t} \quad (31)$$

where  $\omega_q^2 = \vec{q}^2 + m^2$  is the relativistic energy momentum relation. In natural units  $\hbar = 1$  so  $E = \hbar\omega = \omega$ . It may be noted that there are two solutions corresponding to the two square roots  $\omega_{\vec{q}} = \pm\sqrt{\vec{q}^2 + m^2}$ , the positive and negative energy solutions. The equation for complex conjugate  $\phi^*(\vec{x}, t)$  is

$$(\square + m^2)\phi^*(\vec{x}, t) = 0 \quad (32)$$

### Physical interpretation of Klein Gordon wave function

In order to derive interpretation of the Klein Gordon wave function, we will follow the time honoured procedure of using the equation of continuity.



In order to derive the equation of continuity, we multiply (29) by  $\phi^*(\vec{x}, t)$  and (32) by  $\phi(\text{vecx}, t)$  and subtract to get

$$\phi^*(\vec{x}, t) \left\{ \frac{d^2}{dt^2} \phi(\vec{x}, t) - \nabla^2 \phi(\vec{x}, t) \right\} - \left\{ \frac{d^2}{dt^2} \phi(\vec{x}, t) - \nabla^2 \phi(\vec{x}, t) \right\} \phi^*(\vec{x}, t) \quad (33)$$

The above equation can be cast in the formula

$$\partial_0 \left\{ \phi^*(\vec{x}, t) \overleftrightarrow{\partial}_0 \phi(\vec{x}, t) \right\} - \nabla \cdot \left\{ \phi^*(\vec{x}, t) \overleftrightarrow{\nabla} \phi(\vec{x}, t) \right\} = 0. \quad (34)$$

We have the continuity equation

$$\frac{\partial \rho(\vec{x}, t)}{\partial t} + \nabla \cdot \vec{j}(\vec{x}, t) = 0, \quad (35)$$

where we have defined

$$\rho(\vec{x}, t) = i \phi^*(\vec{x}, t) \overleftrightarrow{\partial}_0 \phi(\vec{x}, t) \quad (36)$$

$$\text{and } \vec{j}(\vec{x}, t) = -i \phi^*(\vec{x}, t) \overleftrightarrow{\nabla} \phi(\vec{x}, t). \quad (37)$$

**Question:** Why do we have a factor of  $i$  in the expressions for probability density  $\rho(\vec{x}, t)$  and  $\vec{j}(\vec{x}, t)$ ?

**Ans:** A factor of  $i$  is included so that the two quantities may be real.

The equation of continuity implies that the total probability

$$\int d^3x \rho(\vec{x}, t) = \text{independent of time.} \quad (38)$$

Therefore, we can normalise the solutions by choosing this constant to be one. The plane wave solutions will have to be normalized to Dirac delta function

$$i \int d^x f_{\vec{q}}(\vec{x}, t) \overleftrightarrow{\partial}_0 f_{\vec{p}}(\vec{x}, t) = 2\omega_{\mathbf{q}} \delta(\vec{q} - \vec{p}). \quad (39)$$

Note that the left hand side is zeroth component of four vector current  $j_\mu = i \phi^*(x) \overleftrightarrow{\partial}_\mu \phi(x)$ . A factor of  $2\omega_{\mathbf{q}}$  has been included in the normalisation so as to make the normalisation Lorentz invariant.

**Negative energy solutions** The presence of negative energy solutions gives rise to problems. Very briefly we mention two difficulties.

The expression for probability density is not positive definite. This is easily seen by computing *rho* for the negative energy plane wave solutions.

Note that for a particle with momentum  $\vec{q}$  having momentum  $\vec{q}$  can have arbitrary negative energy ( $= -\sqrt{\vec{q}^2 + m^2}$ ), with no lower limit.

Therefore, in presence of interactions, a particle can radiate energy continuously by making transitions to negative energy states. Since energy has no lower bound, this process will be a never ending process.

Dirac attempted to solve these problems by proposing an equation linear in space and time derivatives. Dirac theory successfully predicted several observations such as spin, magnetic moment and fine structure of H atom. With negative energy states assumed to be all filled, existence of positrons was a brilliant prediction. However, problems of a consistent interpretation, as relativistic quantum mechanics of a point particle, remained unresolved.

It may be emphasized here that several works starting with work of Sudarshan showed that a relativistic theory of a system with finite degrees of freedom has to be free theory. Thus systems with infinite number of degrees of freedom are essential to have interacting relativistic theory.

The second quantisation of Klein Gordon equation and Dirac equation, for example, provides a framework in which a resolution of problems and a consistent interpretation was possible. The quantum field theory of electrons and photons and, later, unified electro-weak model have been brilliantly confirmed by the experiments.

We will discuss the second quantisation of Klein Gordon and Dirac equation in later lectures. In this lecture, we briefly recall the main points of second quantisation of the Schrodinger theory that was covered in a course last semester.

## §2 Quantisation of Klein Gordon Equation

### §1 Important Remarks

We have attempted to follow

**Gasiorowicz S.**, *Elementary Particle Physics*, John Wiley and Sons New York (1966)

as closely as possible taking what can be done in the course keeping the announced goals in mind.

We will refer to portions from Gasiorowicz which are scanned and reproduced. It is strongly recommended that the reader go through the original book as this course progress and try to assimilate what has been left out.

We do not make any attempt, whatsoever, to '*modify*' or '*improve*' what is available in the book.

Frequently, the presentation will be interrupted by the teacher's commentary and some **Gray Boxes** containing short comments, or questions etc. These interruptions are meant to serve two purposes. The purpose is to relate contents from Gasiorowicz to what the students already know students, and to present it as smoothly and as coherently as possible. The **Gray Boxes** are meant to encourage the students to ponder over the questions and comments in the **Gray Boxes** and discuss the same among themselves.

Further detailed discussion of each of these **Gray Boxes** will be provided separately.

## §2 Notation

The symbol  $\overset{\leftrightarrow}{\partial}_\mu$  appears in the combination

$$a \overset{\leftrightarrow}{\partial}_\mu b \equiv a(\partial_\mu b) - (\partial_\mu a)b$$

We use + for hermitian conjugate, when operators are considered, and \* for complex conjugation of ordinary functions.

Integrals over four-dimensional volumes in space-time or momentum space are denoted by

$$\int dx$$

and

$$\int dp$$

unless danger of confusion requires  $\int d^4x$  or  $\int d^4p$ . Three-dimensional integrals will always be denoted by

$$\int d^3x$$

and

$$\int d^3p$$

The notation

$$d \equiv \frac{da}{dt} \equiv \frac{da}{dx^0}$$

will sometimes be used to save space.

State vectors will be denoted by  $\Psi$  or  $\Phi$ .

Finally, we use natural units, in which

$$\hbar = c = 1$$

The symbol  $e_{ijk}(i, j, k) = 1, 2, 3$  stands for the totally antisymmetric tensor

$$\begin{aligned} e_{ijk} &= 1 \text{ when } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ &= -1 \text{ when } i, j, k \text{ is an odd permutation of } 1, 2, 3 \\ &= 0 \text{ otherwise} \end{aligned}$$

$\epsilon_{\alpha\beta\gamma\delta}$  with  $(\alpha, \beta, \gamma, \delta) = 0, 1, 2, 3$  is 4-dimensional generalization of this, with  $\epsilon_{0123} = 1$ .

*Note.* The word *lepton* has not been defined in the text. It refers to the *electron* (mass 0.511 MeV, spin  $\frac{1}{2}$ ), *muon* (mass  $\sim 105$  MeV, spin  $\frac{1}{2}$ ), and the *neutrinos*.

### §3 Relativistic quantum mechanics

The relativistic energy momentum relation is

$$E^2 = p^2 c^2 + m^2 c^4 \quad (40)$$

We will use natural units  $\hbar = 1, c = 1$ . Sometimes I may retain  $\hbar$  factors. TO quantize we use the correspondence

$$\vec{p} \longrightarrow -i\hbar\nabla, \quad i\hbar\frac{\partial\phi(\vec{x}, t)}{\partial t} = H\phi(\vec{x}, t) \quad (41)$$

Eq.(40) and Eq.(41) imply that the wave function obeys

$$\hbar^2\left(\frac{d^2}{dt^2} - \nabla^2\right)\phi(\vec{x}, t) + m^2 c^4 \phi(\vec{x}, t) = 0. \quad (42)$$

Due to appearance of square root in the Hamiltonian, the equation with second order time derivative. has been written down.

This equation, now known as Klein Gordon equation, was originally, Schrodinger derived by Schrodinger. He discarded it when he found that predicted H atom energy levels do not agree with experimental data for H atom. Later he realised that in nonrelativistic limit the result agreed with the Balmer formula.

What are the problems in using Klein Gordon equation as relativistic quantum mechanical wave equation? Where can we find a brief historical account?

It was rediscovered and used for quantum field theory of bosons by Klein and Gordon. The essential new idea here is that the states of a relativistic particle in quantum mechanics are described by  $\phi(\vec{x}, t)$  and the wave function  $\phi(\vec{x}, t)$  is reinterpreted as a set of generalised, one for each space point. The Klein Gordon equation is now treated as describing a classical wave field, just like Maxwell's equations describe the electromagnetic field.

The rules of canonical quantisation procedure can now be applied by reinterpreting Klein Gordon equation as field equation . Its *second quantisation* led to field theory of scalar (spin zero) bosons.

The same reinterpretation and second quantisation has been carried out for Dirac equation and the Schrodinger equation even though latter led to a consistent theory.

The quantum theory of fields was created by the application of a form of the Heisenberg quantization rules to the classical electromagnetic field.<sup>1</sup> There are special complications which arise in that case, caused in part by the fact that the electromagnetic field has several components, and in part by the fact that electromagnetic waves move with the velocity of light. The basic concepts of field quantization are more easily studied in connection with the scalar field, which has only one component and which will be denoted by  $\phi(x)$ . The field  $\phi(x)$  is assumed to have a Fourier transform

$$\phi(x) = \frac{1}{(2\pi)^{3/4}} \int d^3q e^{-iqx} \tilde{\phi}(q) \quad (1.1)$$

If this field is to describe the motion of a packet whose frequency and wave number are related by

$$q^0 = \sqrt{(\mathbf{q}^k)^2 + \mu^2} = \sqrt{\mathbf{q}^2 + \mu^2} \equiv \omega_{\mathbf{q}} \quad (1.2)$$

which, using the De Broglie relations between wave and particle attributes, describes the relation between the energy and momentum for a particle of mass  $\mu$ , then  $\tilde{\phi}(q)$  must have the form

$$\begin{aligned} \tilde{\phi}(q) &= \delta(q_\mu q^\mu - \mu^2) \chi(q) \\ &= \delta(q^2 - \mu^2) \chi(q) \end{aligned} \quad (1.3)$$

Thus the field  $\phi(x)$  must satisfy the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + \mu^2) \phi(x) \equiv (\square + \mu^2) \phi(x) = 0 \quad (1.4)$$

<sup>1</sup> The historical development of field theory may be followed through the collection of papers published under the title *Quantum Electrodynamics*, J. Schwinger (ed.), Dover, New York (1958).

For our purposes it will be enough to consider plane wave solutions of this equation. We shall use the notation

$$f_{\mathbf{q}}(x) = \frac{1}{(2\pi)^{3/2}} e^{-iqx} \quad (1.5)$$

where

$$\begin{aligned} qx &= q^\mu x_\mu = q^0 x_0 - \mathbf{q} \cdot \mathbf{x} \\ q^\mu &= (\omega_{\mathbf{q}}, \mathbf{q}) \end{aligned} \quad (1.6)$$

The plane wave solutions will always be understood as the limiting case of wave packet solutions, so that we will allow ourselves integrations by parts of the spatial coordinates. The solutions (1.5) satisfy the following normalization condition

$$\begin{aligned} i \int d^3x \left[ f_{\mathbf{q}'}^*(x) \frac{\partial}{\partial x_0} f_{\mathbf{q}}(x) - \frac{\partial}{\partial x_0} f_{\mathbf{q}'}^*(x) f_{\mathbf{q}}(x) \right] \\ = i \int d^3x f_{\mathbf{q}'}^*(x) \overset{\leftrightarrow}{\partial} f_{\mathbf{q}}(x) = 2\omega_{\mathbf{q}} \delta(\mathbf{q} - \mathbf{q}') \end{aligned} \quad (1.7)$$

Also

$$\int d^3x f_{\mathbf{q}}(x) \overset{\leftrightarrow}{\partial} f_{\mathbf{q}}(x) = \int d^3x f_{\mathbf{q}'}^*(x) \overset{\leftrightarrow}{\partial} f_{\mathbf{q}'}^*(x) = 0 \quad (1.8)$$

We shall call the solutions  $f_{\mathbf{q}}(x)$  positive frequency solutions and the solutions  $f_{\mathbf{q}'}^*(x)$  negative frequency solutions.



In contrast to the functions that satisfy the Schrödinger equation,  $|\phi(x)|^2$  may not be used as a density, for in general the integral of  $|\phi(x)|^2$  over all space is not independent of time. A suitable density may be found with the help of the observation that the four-current

$$j^\mu(x) = i \phi^*(x) \overleftrightarrow{\partial}_\mu \phi(x) \equiv i \phi^*(x) \overleftrightarrow{\partial}^\mu \phi(x) \quad (1.9)$$

satisfies the condition

$$\begin{aligned} \partial_\mu j^\mu(x) &= i \partial_\mu (\phi^*(x) \overleftrightarrow{\partial}^\mu \phi(x) - \overleftrightarrow{\partial}^\mu \phi^*(x) \phi(x)) \\ &= i(\phi^*(x) \square \phi(x) - \square \phi^*(x) \phi(x)) = 0 \end{aligned} \quad (1.10)$$

Hence, with  $j^\mu(x) = (\rho(x), \mathbf{j}(x))$ , we get, on integrating by parts,

$$\int d^3x \partial^0 \rho(x) = - \int d^3x \nabla \cdot \mathbf{j}(x) = 0$$

so that

$$\int d^3x \rho(x) = \text{constant.}$$

Scan-5

Take a break

Why can we not continue using the same normalisation as in the Schrodinger case? What is the need to derive equation of continuity again?

## §4 Recall canonical quantisation

When we think of quantisation of system, we already have in mind a classical description of the system in mind. This may not always be possible, for example, interacting spins is one such system.

In canonical quantisation the starting point is the Lagrangian formulation of dynamics. A state of the system is completely described by a set of coordinate  $\{q_1, q_2, \dots\}$  and generalised velocities,  $\{\dot{q}_1, \dot{q}_2, \dots\}$ , The dynamics of the system is governed by a Lagrangian,  $\mathcal{L}(q, \dot{q})$ , which is a function of the generalised coordinates and velocities.

A transition to the Hamiltonian form of dynamics is made by introducing the canonical momenta,  $p_i$ , conjugate to the generalised coordinates defined by

$$p_k = \frac{\partial}{\partial \dot{q}_k} \mathcal{L} \quad (43)$$

The canonical quantisation consists in assuming that the generalised coordinates and momenta are to be treated as hermitian operators in some Hilbert state satisfying the canonical commutation relations

$$[q_i, p_j] = i\hbar\delta_{ij}. \quad (44)$$

The states of the system are described vectors in the Hilbert space.

All this is gives us only a mathematical frame work.  
Where is the physical interpretation? What is the Hilbert state? How it gets related to the experimental observations.

A complete discussion will require going details of postulates of quantum theory. Instead of digressing to that, here we will describe the process in a form that is suitable for in our present context.

The next step in discussion of quantised theory is to find a complete set of commuting observables constructed out of coordinates and momenta gives a complete orthonormal set. The linear span of this complete orthonormal set is the Hilbert space of quantum states.

At this stage it is important to realise that there is no unique commuting set of on observables. Different sets lead to different representaions. To fix our idea in quantum mechanics of a point particle two such sets are  $\{\vec{r}\}$  and  $\{\vec{p}\}$ . the position operators and momentum operators. These lead to two different representations known as the coordinate and the momentum representation.

The states of simultaneous eigenvectors have a useful physical interpretation. Each dynamical variable in the commuting set has a definite value in the states corresponding to the simultaneous eigenvectors.

For any other state  $|\Psi\rangle$ , the probability amplitudes of the dynamical variables having different sets of values is obtained by expanding of the state vectors  $|\Psi\rangle$  in terms of the basis of simultaneous eigenvectors and noting down the coefficients.

Thus once we know the commuting set of observations, a Hilbert space and physical interpretation can be built up following the procedure outlined above.

## §5 Over to Gasirowicz

We start with the canonical commutation relations. The basic equations of canonical quantization procedure are as follows.

In ordinary quantum mechanics these operators act in a vector space (Hilbert space of square integrable functions), and they are assumed to obey equations of motion which formally look exactly like the classical equations of motion. In order to establish the quantum theory which corresponds (in the above sense) to the classical scalar field we must determine (1) what is the momentum canonically conjugate to the "variable"  $\phi(\mathbf{x}, t)$  and (2) what is the generalization of the Heisenberg commutation rules applicable to this system. We follow the procedure in the mechanical case, which is to find a Lagrangian that leads to the equations of motion (1.4) and from it to obtain the canonical momentum and the Hamiltonian. The equations below show the parallelism between a system of  $N$  free particles and the scalar field. The particle Lagrangian

$$L(t) = \frac{1}{2} m \sum_{i=1}^{3N} (\dot{q}_i(t))^2$$

contains a sum over the labels for each degree of freedom. If we recognize that the variable  $\mathbf{x}$  in  $\phi(\mathbf{x}, t)$  is also a labeling over a continuously varying index [a more transparent notation might be  $\phi_x(t)$ ], we see that the field Lagrangian must be an integral over a Lagrangian density

$$L(t) = \int d^3x \mathcal{L}(\mathbf{x}, t) \quad (1.11)$$

The equations of motion for the mechanical system are obtained from

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i(t)} \right) = \frac{\partial L}{\partial q_i(t)} \quad (i = 1 \cdots 3N)$$

which follow from the principle of least action. The corresponding field equations are

$$\partial^\mu \frac{\partial \mathcal{L}(\phi(x), \partial^\alpha \phi(x))}{\partial (\partial^\mu \phi(x))} = \frac{\partial \mathcal{L}}{\partial \phi(x)} \quad (1.12)$$

The canonical momentum for the particle system is

$$p_i(t) = \frac{\partial L}{\partial \dot{q}_i(t)}$$

and the corresponding quantity for the field is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial (\partial^0 \phi(x))} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \quad (1.13)$$

Finally, the Hamiltonian is defined by

$$H = \sum_{i=1}^{N} p_i(t) \dot{q}_i(t) - L(t)$$

and the field Hamiltonian is

$$H = \int d^3x \mathcal{H}(x) \quad (1.14)$$

with

$$\mathcal{H}(x) = \pi(x) \partial^0 \phi(x) - \mathcal{L}(x) \quad (1.15)$$

The Heisenberg commutation relations

$$\begin{aligned} [p_i(t), q_j(t)] &= -i \delta_{ij} \\ [p_i(t), p_j(t)] &= [q_i(t), q_j(t)] = 0 \end{aligned}$$

have as their field counterparts

$$[\pi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = -i \delta(\mathbf{x} - \mathbf{y}) \quad (1.16)$$

will also be satisfied. The field  $\phi^*(x)$  will have its canonically conjugate momentum

$$\pi^*(x) = \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi^*(x))}$$

The Hamiltonian density will be

$$\mathcal{H}(x) = \pi(x) \partial^0 \phi(x) + \pi^*(x) \partial^0 \phi^*(x) - \mathcal{L}$$

and the additional commutation relation

$$[\pi^*(\mathbf{x}, t), \phi^*(\mathbf{y}, t)] = -i \delta(\mathbf{x} - \mathbf{y})$$

will be assumed to hold. All commutators involving starred with unstarred fields vanish at equal times, since these are independent fields.

It is important to note that the commutation relations are only defined for equal times. Once these are given, the commutation relations at different times are determined by the development of the system in time,

## §6 Number Operators

In quantum field theory the field operator does not have a direct physical interpretation. Connection with physical quantities is arrived at by expanding the field operators in terms of plane wave solutions. The expansion coefficients  $A, A^\dagger, B, B^\dagger$  (see (1.37) below) will be operators. The commutations relations for these operators will be obtained by first writing them in terms of  $\phi(x), \partial_0 \phi(x)$  etc (see 1.38); next we note make use of ETCR (1.16) remembering that (1.13) gives  $\partial_0 \phi^*(x) = \pi^0(x)$ . Now go through the steps presented below from **Gasiorowicz**. The equations (1.39) and (1.40) give the final results for all noncommuting operators.

$$\begin{aligned}\phi(x) &= \int \frac{d^3q}{2\omega_q} [f_q(x) A(q) + f_q^*(x) B^*(q)] \\ \phi^*(x) &= \int \frac{d^3q}{2\omega_q} [f_q(x) B(q) + f_q^*(x) A^*(q)]\end{aligned}\quad (1.37)$$

with  $f_q(x)$  as defined in (1.5). The orthonormality conditions (1.7) and (1.8) may be used to invert these expansions. We get

$$\begin{aligned}A(q) &= i \int d^3x f_q^*(x) \overleftrightarrow{\partial}^0 \phi(x) \\ B(q) &= i \int d^3x f_q^*(x) \overleftrightarrow{\partial}^0 \phi^*(x) \\ A^*(q) &= i \int d^3x \phi^*(x) \overleftrightarrow{\partial}^0 f_q(x) \\ B^*(q) &= i \int d^3x \phi(x) \overleftrightarrow{\partial}^0 f_q(x)\end{aligned}\quad (1.38)$$

The commutation relations satisfied by the operators  $A(q), \dots, B^*(q)$  are easily obtained. For example,<sup>1</sup>

$$\begin{aligned}[A(q), A^*(q')] &= \int d^3x \int d^3y [f_q^*(x) \overleftrightarrow{\partial}^0 \phi(x), f_{q'}(y) \overleftrightarrow{\partial}^0 \phi^*(y)] \\ &= \int d^3x \int d^3y \{-f_q^*(x) f_{q'}(y) [\phi(x), \phi^*(y)] \\ &\quad - f_q^*(x) f_{q'}(y) [\phi(x), \phi^*(y)]\} + \dots\end{aligned}$$

We choose  $x_0 = y_0$  and use the equal time commutation to get

$$\begin{aligned}[A(q), A^*(q')] &= i \int d^3x (f_q^*(x) f_{q'}(x) - f_q^*(x) f_{q'}(x)) \\ &= 2\omega_q \delta(\mathbf{q} - \mathbf{q}')\end{aligned}\quad (1.39)$$

Similarly, we find that

$$[B(q), B^*(q')] = 2\omega_q \delta(\mathbf{q} - \mathbf{q}')\quad (1.40)$$

with all other pairs of operators commuting.

Verify equations (1.38) to (1.40) given in the book by Gasiorowicz.

Verify expression (1.47) for the total Hamiltonian given by Gasiorowicz.

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This follows from Lorentz invariance and the fact that a spacelike  $x$  can be rotated into  $-x$ . Hence to satisfy (1.45) we must have

$$\Delta(x) = 0 \quad x^2 < 0 \quad (1.46)$$

Thus the commutator  $[\phi(x), \phi^*(y)]$  vanishes when the separation between the points is spacelike. We shall have more to say about this important result.

Let us return to the diagonalization of  $H$ . Some algebraic manipulations give us  $H$  in terms of the operators  $A(q), \dots, B^+(q)$ ,

$$H = \int \frac{d^3q}{2\omega_q} [\omega_q A^+(q) A(q) + \omega_q B(q) B^+(q)] \quad (1.47)$$

From this we find, with the help of the commutation relations (1.39) and (1.40), that

$$[A(q), H] = \omega_q A(q) \quad (1.48)$$

and

$$[A^+(q), H] = -\omega_q A^+(q) \quad (1.49)$$

Similarly

$$[B(q), H] = \omega_q B(q) \quad (1.50)$$

and

$$[B^+(q), H] = -\omega_q B^+(q) \quad (1.51)$$

If the state  $\Psi_E$  is an eigenstate of  $H$  with energy  $E$ , then

$$H A(q)\Psi_E = A(q)H\Psi_E + [H, A(q)]\Psi_E = (E - \omega_q)A(q)\Psi_E \quad (1.52)$$

Similarly

$$H A^+(q)\Psi_E = (E + \omega_q)A^+(q)\Psi_E \quad (1.53)$$

and analogously for the states  $B(q)\Psi_E$  and  $B^+(q)\Psi_E$ . Thus the state  $A(q)\Psi_E$  is again an eigenstate of  $H$ , but with energy lowered by an amount  $\omega_q$ . The state  $A^+(q)\Psi_E$  is also an eigenstate of  $H$ , but with energy raised by  $\omega_q$ ; similar statements hold for  $B(q)\Psi_E$  and  $B^+(q)\Psi_E$ . We may speak of  $A(q)$  and  $B(q)$  as *annihilation operators*, for they take away energy from the system, and  $A^+(q)$  and  $B^+(q)$  as *creation operators*, for they add energy to the system. The Hamiltonian is a positive definite operator, and there must be a state of lowest energy. By adding a constant to  $H$ , if necessary, we may shift the energy scale so that the lowest energy is zero. We assume

## Number operators and the Hilbert space

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In the problem of a single quantum harmonic oscillator the operators  $a, a^\dagger$  and  $N = a^\dagger a$  are defined which obey the commutation relations

$$[a, a^\dagger] = 1, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \quad (45)$$



The eigenvalues of  $N$  turned out to be all nonnegative integers  $0, 1, 2, \dots$  and the operators  $a, a^\dagger$  act as lowering and raising operators for the number operator.

Here we have a similar situation. Number operators  $N(q), N(q')$  are hermitian and  $N(q), N(q')$  commute pairwise if  $q \neq q'$ . Therefore these operators form a commuting set and have simultaneous eigenvectors which form a complete orthonormal set. Their linear span will give the Hilbert space. There exists a state called *itvacuum* and denoted by  $|0\rangle$  such that all the number operators which is eigenvector of all number operators with zero eigenvalue.

$$N(\vec{q})|0\rangle = 0, \quad \text{for all } \vec{q}. \quad (46)$$

## §7 Physical interpretation

Let us recall some results from second quantised Schrodinger field theory. The Schrodinger wave function was treated as classical field and was expanded in terms of eigenfunctions  $u_n(\vec{x})$

$$\psi(\vec{x}, t) = \sum_n a_n u_n(\vec{x}) e^{-iE_n t/\hbar} \quad (47)$$

of the total Hamiltonian  $H = \frac{p^2}{2m} + V(\vec{r})$

The canonical quantisation led to the following commutation relations for operators  $a_k, a_k^\dagger$  and  $N_k = a_k^\dagger a_k$ .

$$[a_k, a_m^\dagger] = \delta_{km}, \quad [N_k, a_k] = -a_k, \quad [N_k, a_k^\dagger] = a_k^\dagger \quad (48)$$

There is a close resemblance between the above equations and the corresponding equations for the scalar field. So we can carry over the interpretation of the Schrodinger theory to the Klein Gordon case. The only way the equations for the Klein Gordon case differ from our discussion of quantised Schrodinger field is that the discrete index the operators  $A(q), B(q)$  and  $N(q)$  in scalar field case depend on a continuous index.

So the interpretation of the number operators  $N(q)$  is that it gives the *number density*, *i.e.* the number of particles in range  $\vec{q}$  and  $\vec{q} + d\vec{q}$  equals  $N(\vec{q})d\vec{q}$ .

So read on Gasiorowicz where all this has and more has been explained.

§8 Back to Gasiorowicz

Let us return to the diagonalization of  $H$ . Some algebraic manipulations give us  $H$  in terms of the operators  $A(q), \dots, B^+(q)$ ,

$$H = \int \frac{d^3q}{2\omega_q} [\omega_q A^+(q) A(q) + \omega_q B(q) B^+(q)] \quad (1.47)$$

From this we find, with the help of the commutation relations (1.39) and (1.40), that

$$[A(q), H] = \omega_q A(q) \quad (1.48)$$

and

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If the state  $\Psi_E$  is an eigenstate of  $H$  with energy  $E$ , then

$$H A(q)\Psi_E = A(q)H\Psi_E + [H, A(q)]\Psi_E = (E - \omega_q)A(q)\Psi_E \quad (1.52)$$

Similarly

$$H A^+(q)\Psi_E = (E + \omega_q)A^+(q)\Psi_E \quad (1.53)$$

and analogously for the states  $B(q)\Psi_E$  and  $B^+(q)\Psi_E$ . Thus the state  $A(q)\Psi_E$  is again an eigenstate of  $H$ , but with energy lowered by an amount  $\omega_q$ . The state  $A^+(q)\Psi_E$  is also an eigenstate of  $H$ , but with energy raised by  $\omega_q$ ; similar statements hold for  $B(q)\Psi_E$  and  $B^+(q)\Psi_E$ . We may speak of  $A(q)$  and  $B(q)$  as *annihilation operators*, for they take away energy from the system, and  $A^+(q)$  and  $B^+(q)$  as *creation operators*, for they add energy to the system. The Hamiltonian is a positive definite operator, and there must be a state of lowest energy. By adding a constant to  $H$ , if necessary, we may shift the energy scale so that the lowest energy is zero. We assume now that there is a *unique* state of zero energy, denoted by  $\Psi_0$  and called the *vacuum state*. It must satisfy

$$H\Psi_0 = 0 \quad (1.54)$$

and

$$A(q)\Psi_0 = B(q)\Psi_0 = 0 \quad \text{for all } q. \quad (1.55)$$

We shall normalize it so that

$$(\Psi_0, \Psi_0) = 1 \quad (1.56)$$

The vacuum expectation value of  $H$ , with the help of (1.55), is infinite. Thus to shift the energy scale we must add an infinite constant or, alternately, make use of our freedom to reorder the operators in  $H$ . If we define  $H$  as

$$H = \int \frac{d^3q}{2\omega_q} [\omega_q A^+(q) A(q) + \omega_q B^+(q) B(q)] \quad (1.57)$$

there are no difficulties. The fact that the additive constant is infinite is the first manifestation of trouble we encounter in a much less benign form later: frequently matrix elements of products of operators at the same point lead to infinities, and only sometimes will we be able to get rid of them. To make sure that the most trivial infinities do not occur, we shall always define quantities given in terms of free field operators (such as  $H$ ) as *normal ordered*, i.e., as decomposed into free creation and annihilation operators and then reordered so that the annihilation operators stand to the right of the creation operators. This instruction is denoted by colons. Thus  $H$ , as it appears in (1.57), is written

$$H = : \int d^3x [\pi(x)\pi^*(x) + \nabla \phi(x) \cdot \nabla \phi^*(x) + \mu \overline{\phi(x)} \phi^*(x)] : \quad (1.58)$$

in  $x$ -space.

Consider now the states

$$\Psi_{a_A} \equiv A^+(q)\Psi_0 \quad (1.59)$$

and

$$\Psi_{a_B} \equiv B^+(q)\Psi_0 \quad (1.60)$$

In both cases we find that the states have energy  $\omega_q$ :

$$\begin{aligned} H\Psi_{a_A} &= HA^+(q)\Psi_0 = [H, A^+(q)]\Psi_0 \\ &= \omega_q A^+(q)\Psi_0 \\ &= \omega_q \Psi_{a_A} \end{aligned} \quad (1.61)$$

The orthonormality properties of the states are

$$\begin{aligned} (\Psi_{a_A}, \Psi_{a_A'}) &= (\Psi_0, A(q')A^+(q)\Psi_0) = (\Psi_0, [A(q'), A^+(q)]\Psi_0) \\ &= 2\omega_q \delta(\mathbf{q} - \mathbf{q}') \end{aligned} \quad (1.62)$$

Similarly,

$$(\Psi_{a_B}, \Psi_{a_B'}) = 2\omega_q \delta(\mathbf{q} - \mathbf{q}') \quad (1.63)$$

and

$$(\Psi_{a_A}, \Psi_{a_B}) = 0 \quad (1.64)$$

A generalization of the states (1.59) and (1.60) is

$$\Psi_{q_1 \dots q_m; k_1 \dots k_n} = \frac{1}{\sqrt{m! n!}} A^+(q_1) \dots A^+(q_m) B^+(k_1) \dots B^+(k_n) \Psi_0 \quad (1.65)$$

The energy of the state is calculated as follows

$$\begin{aligned} & H A^+(q_1) A^+(q_2) \dots B^+(k_n) \Psi_0 \\ &= ([H, A^+(q_1)] + A^+(q_1) H) A^+(q_2) \dots B^+(k_n) \Psi_0 \\ &= \omega_{q_1} A^+(q_1) \dots B^+(k_n) \Psi_0 + A^+(q_2) H A^+(q_1) \dots B^+(k_n) \Psi_0 \\ &= \omega_{q_1} A^+(q_1) \dots B^+(k_n) \Psi_0 + A^+(q_2) A^+(q_1) H \dots B^+(k_n) \Psi_0 \\ &\quad + A^+(q_2) [H, A^+(q_1)] \dots B^+(k_n) \Psi_0 \\ &= (\omega_{q_1} + \omega_{q_2}) A^+(q_1) \dots B^+(k_n) \Psi_0 + A^+(q_2) A^+(q_1) H \dots B^+(k_n) \Psi_0 \\ &= \dots \\ &= (\omega_{q_1} + \omega_{q_2} + \dots + \omega_{k_n}) A^+(q_1) \dots B^+(k_n) \Psi_0 \end{aligned} \quad (1.66)$$

The energy is therefore the sum of the energies created by the various operators. This suggests the interpretation that  $A^+$  and  $B^+$  create quanta carrying energy. To test this interpretation we consider the operator

$$\mathcal{N}_A(q) = \frac{1}{2\omega_q} A^+(q) A(q) \quad (1.67)$$

which appears in  $H$  and in  $P$ . With the help of the commutation relations we can show that

$$\mathcal{N}_A(q) \Psi_{q_1, \dots, k_1, \dots} = [\delta(\mathbf{q} - \mathbf{q}_1) + \dots + \delta(\mathbf{q} - \mathbf{q}_m)] \Psi_{q_1, \dots, k_1, \dots} \quad (1.68)$$

Thus the states  $\Psi_{q_1, \dots, k_1, \dots}$  are eigenstates of  $\mathcal{N}_A(q)$ . Moreover, the operator

$$N_A(\Omega) = \int_{\Omega} d^3q \mathcal{N}_A(q) \quad (1.69)$$

has the eigenvalue  $\nu$ , where  $\nu$  is the number of vectors among  $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$  which lie in the volume  $\Omega$  in momentum space. We may interpret  $\mathcal{N}_A$  as a *number density* operator, the number referring to the number of quanta created by the operator  $A^+(q)$ . Similarly

$$\mathcal{N}_B(q) = \frac{1}{2\omega_q} B^+(q) B(q) \quad (1.70)$$

is the number density operator for the quanta created by the operator  $B^+(q)$ . The Hamiltonian may now be rewritten in the form

$$H = \int d^3q [\omega_q \mathcal{N}_A(q) + \omega_q \mathcal{N}_B(q)] \quad (1.71)$$

Thus the energy in a volume element  $d^3q$  in momentum space is  $\omega_q$  times the number of quanta of types  $A$  and  $B$  in that volume of momentum space. Simple manipulations show that the operator  $\mathbf{P}$ , defined in (1.29) may be written as

$$\mathbf{P} = \int d^3q [\mathbf{q} \mathcal{N}_A(q) + \mathbf{q} \mathcal{N}_B(q)] \quad (1.72)$$

If we accept the quantum interpretation, this expression shows that we may identify  $\mathbf{P}$  with the momentum operator. It has the correct additivity properties. If we apply  $H$  and  $\mathbf{P}$  to a one particle state of type  $A$  or  $B$ , we obtain the eigenvalues  $\omega_q$  and  $\mathbf{q}$ , respectively,

$$\begin{aligned} H\Psi_q &= \omega_q\Psi_q \\ \mathbf{P}\Psi_q &= \mathbf{q}\Psi_q \end{aligned}$$

from which we deduce that both quanta of type  $A$  and  $B$  have mass  $\mu$ .

One can also show that the Klein Gordon equation describes a zero spin particle. This is done by showing that under rotations the wave function of one particle at rest does not change hence its spin angular momentum is zero. For more detail we refer you to the original book by Gasiorowicz.

This completes our discussion of physical interpretation of quantised Klein Gordon field.