ME-02 Lessons in Physics*

Summation Convention, ϵ, δ symbols and All That

A. K. Kapoor
http://0space.org/users/kapoor
akkapoor@cmi.ac.in; akkhcu@gmail.com

Contents

§1 Learning Goals

⇑–§1–§2–§3–⇓

You will learn about Einstein summation convention, Kronecker delta symbol and Levi-Civita epsilon symbol. Examples of usage of Kronecker delta and Levi-Civita symbols to vector algebra are presented.

§2 Summation Convention

⇑–§1–§2–§3–⇓

§2.1 Einstein Summation Convention

We describe the Einstein summation convention and give some examples.

1. **Summation convention**
   If $\vec{x} = (x_1, x_2, x_3)$ is vector, square of its length is given by

$$|\vec{x}|^2 = \sum_{i=1}^{3} x_i^2.$$

   We can rewrite it as

$$|\vec{x}|^2 = \sum_{i=1}^{3} x_i x_i.$$

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In this form the index i is repeated and is summed over all values. The *Einstein summation convention* says all repeated indices are automatically summed over all possible values. With this convention we write

$$|\vec{x}|^2 = x_i x_i.$$

2. **Dummy index**
   The index which is summed over all values is called a *dummy index.* A dummy index can be replaced with any other index taking the same set of values. Thus we can write $|\vec{x}|^2$ as $x_i x_i$, or as $x_j x_j$. Obviously the two expressions are equal.

3. **Free index must balance**
   An index which appears only once in an expression is not summed, is called a *free index.* Every term of an equation (or an expression) the free indices must balance.

4. **A relation having having a free index**
   If an index appears as a free index in an equation, it is understood, by convention, that the hold for all values of the free index. As an example, matrix multiplication of a column vector u by a matrix, $v = Au$, is normally written as

   $$v_i = \sum_{j=1}^{N} A_{ij} u_j, i = 1, \ldots, N, \tag{1}$$

   With the above convention we will write it as

   $$v_i = A_{ij} u_j \tag{2}$$

   In the above equation $i$is a free index. It is understood that the above equation holds for all values of the free index i.

§2.2 Kronecker Delta and Levi-Civita Symbols

⇑

Convention In this write up we assume Einstein summation convention for repeated indices.

Definition 1 *The Kronecker delta symbol* δ_{ij} *is defined as*

$$\delta_{ij} = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j. \end{cases} \tag{3}$$

Definition 2 *The Levi-Civita symbol* ϵ_{ijk} *(with three indices) is a completely anti-symmetric under exchange of any two indices. So for example*

$$\epsilon_{ijk} = -\epsilon_{jik}; \epsilon_{ijk} = -\epsilon_{ikj}; \epsilon_{kij} = -\epsilon_{kij}.$$

Here the indices ijk *take values from 1 to 3.*

The symbol ϵ_{ijk} has only one independent component and we have $\epsilon_{123} = 1$. All other components are related to ϵ_{123} and turn out to be either zero or ± 1.

The definition of the Levi-Civita is easily generalized to the case of any number of indices. So with N indices $i_1, i_2, ..., i_N$ all taking values $1, 2, ..., N$, we have the symbol $\epsilon_{i_1,i_2,...i_N}$ antisymmetric under exchange of any pair of two indices and $\epsilon_{123..N} = 1$.

)‖(**Short Examples 1** *We explicitly list values of Kronecker delta and epsilon symbols when the indices run from 1 to 3.*

(1a) $\delta_{11} = \delta_{22} = \delta_{33} = 1$

(1b) $\delta_{12} = \delta_{21} = \delta_{23} = \delta_{32} = \delta_{31} = \delta_{13} = 0$

(1c) The six non-zero components of epsilon symbol are

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$

$$\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$$

(1d) All other components of ϵ_{ijk} *vanish when any two indices coincide. So, for example*

$$\epsilon_{111} = \epsilon_{222} = \epsilon_{333} = 0$$

$$\epsilon_{112} = \epsilon_{122} = \epsilon_{133} = ... = 0$$

A useful result *If* f_{ijk} *is any object which is totally antisymmetric in its indices, then it must be proportional to the Levi-Civita symbol. Thus*

$$f_{ijk} = C\epsilon_{ijk}; \text{ and } C = f_{123}$$

Some identities *We give some identities of Kronecker delta and the Levi-Civita symbols for the case when the indices take three values 1,2,3.*

$$\delta_{ii} = 3; \qquad \epsilon_{ijk}\epsilon_{ijk} = 6 \tag{4}$$

For the Levi-Civita symbol we have the following identities.

$$\epsilon_{ijk}\,\epsilon_{ljk} = 2\,\delta_{il} \tag{5}$$

$$\epsilon_{ijk}\,\epsilon_{lmk} = (\,\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}) \tag{6}$$

$$\epsilon_{ijk}\,\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{7}$$

The determinant of a 3×3 *matrix* X *has an expression in terms of Levi-Civita symbol.*

$$\det X = \frac{1}{3!}\epsilon_{ijk}\epsilon_{\ell mn}X_{i\ell}X_{jm}X_{kn}. \tag{8}$$

This result generalizes matrices having any dimension.

§2.3 Examples

Summation convention

)|(**Short Examples 2** *Let S_{ij} and A_{ij} are respectively symmetric and anti-symmetric under exchange ij and T_{ij} be arbitrary second rank tensor. Then*

(2a) $S_{ij}T_{ij} = \frac{1}{2}S_{ij}\Big(T_{ij} + T_{ji}\Big)$

(2b) $A_{ij}T_{ij} = \frac{1}{2}A_{ij}\Big(T_{ij} - T_{ji}\Big)$

(2c) $S_{ij}A_{ij} = 0.$

Proof of (2a) Let S_{ij} be symmetric under exchange of indices $i \leftrightarrow j$ and T_{ij} be arbitrary tensor of rank 2. Thus we are given $S_{ij} = S_{ji}$. We will now show that

$$S_{ij}T_{ij} = \frac{1}{2}\Big(S_{ij}(T_{ij} + T_{ji})\Big).$$

Let σ denote the left hand side, $\sigma = S_{ij}T_{ij}$ Consider

$$\sigma \quad = \quad S_{ij}T_{ij} = S_{ji}T_{ij} \qquad \text{used given symmetry property of S} \tag{9}$$

Now replace dummy indices i, j by a new set mn to get

$$\sigma = S_{ji}T_{ij} \quad = \quad S_{nm}T_{mn} \qquad \text{replaced } i \to m, j \to n \tag{10}$$

$$= \quad S_{ij}T_{ji} \qquad \text{replaced } m \to j, n \to i \tag{11}$$

This implies that the $\frac{1}{2}\Big(S_{ij}(T_{ij}+Tji)\Big) = \frac{\sigma}{2}+\frac{\sigma}{2} = \sigma$. which is the desired result. Proof of (2b) is written along similar lines. For a proof of (2c), use (2a) or (2b).

§2.4 Use of ϵ, δ symbols in vector algebra

The use of Kronecker delta and Levi-Civita epsilon symbols for vector algebra and vector calculus simplifies computations. Here we give a few elementary examples to illustrate usage of these symbols.

[1] The dot product of two vectors $\vec{A} \cdot \vec{B}$ can be written as

$$\vec{A} \cdot \vec{B} = \delta_{jk}A_jB_k \tag{12}$$

[2] The cross product of two vectors $\vec{C} = \vec{A} \times \vec{B}$ can be written as

$$C_i = \epsilon_{ijk}A_jB_k \tag{13}$$

[3] The triple product $[\vec{A}, \vec{B}, \vec{C}]$ can be represented as

$$[\vec{A}, \vec{B}, \vec{C}] = \epsilon_{ijk}A_iB_jC_k \tag{14}$$

[4] Using the above expression is is easy to see that the cross product of a vector with itself vanishes. This is seen as follows. Let $\vec{C} = \vec{A} \times \vec{A}$, then

$$C_i = \epsilon_{ijk}(A_j A_k). \tag{15}$$

Here ϵ_{ijk} is antisymmetric under exchange $j \leftrightarrow k$ whereas $A_j A_k$ is symmetric. Hence the sum over all jk vanishes.

[5] Vector algebra identities can be used to derive identities for the Kronecker delta and Levi-Civita epsilon symbols. For example

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{B} \cdot \vec{C})(\vec{A} \cdot \vec{D})$$

implies

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn}. - \delta_{km}\delta_{jn}$$

The proof of this result is left as an exercise for the reader.

§3 EndNotes

⇑–§1–§2–§3–⇓

1. **Food for your thought**

   (a) Writing out all terms for a 2×2 matrix A, explicitly verify that

   $$\epsilon_{ij}\epsilon_{mn}A_{im}A_{jn} = 2 \det A.$$

   where the indices i, j, m, n take values 1 and 2.

   (b) For a three by three matrix A show that

   $$\det A = \epsilon_{ijk}A_{i1}A_{j2}A_{k3}.$$

2. The discussion of Kronecker δ and summation convention presented here is based on Woodhouse [1] Examples 3.1-3.6.

3. For introduction to Kronecker delta and Levi Civita Symbol and applications to vector calculus and electromagnetic theory see, for example `https://arxiv.org/pdf/1406.3060.pdf`

4. 0space Link for Levi Civita tensor On 0space.org ;
   See also Kronecker Delta function δ_{ij} and Levi-Civita (Epsilon) symbol ϵ_{ijk}

References

[1] Woodhouse N. M. J. *Introduction to Analytical Dynamics*. Springer, London Limited, New edition, 2009.

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