MM-16 Lessons in Vector Algebra

A Quick Review of Vectors

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§1 Lesson Overview

Learning Goals

In this lesson, we shall begin with vectors as geometrical objects. A quick review of a few vector algebra identities will be presented. With a choice of coordinate system, vectors are described as objects with three components. We will present a result on change in components of a vector when coordinate axes are changed.

Prerequisites

A first exposure to vector algebra; Dot, cross and triple products. Components of a vector along coordinate axes.

§2 Vectors as Geometrical Objects

The vectors are introduced geometrical objects having a magnitude and direction. Then one can define various operations on vectors. These include multiplication by a real number, addition of two vectors, taking dot and cross products of two vectors.

A large variety of physical quantities, such as displacement, velocity etc., appear as vectors. The laws of physics are formulated as vector (tensor equation) equations. In order to be able to make numerical predictions and to compare them with experimental data, geometric description of vector physical quantities turns out inadequate,if not useless. While the orbit of a planet around the Sun can be geometrically described as ellipses, but to use laws of physics to make predictions and detailed numerical comparisons observations it is essential to introduce a coordinate system and work with the three components of the position vector.

Notation & Convention:
We shall use boldface letters, $\mathbf{A}, \mathbf{B}, \mathbf{C}$.., to denote vectors.
If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the coordinate axes, a given vector can be expressed as a linear combination of these unit vectors along the three axes.

$$\mathbf{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}. \tag{1}$$

Here A_x, A_y, A_z are the components of vectors in chosen set of axes. We will use the notation $\vec{A} = (A_x, A_y, A_z)$ to denote the set of the three components of a vector.
Also, we shall use $\widetilde{\mathsf{A}}$ to denote the of components of a vector $\mathbf{A}$ as the column vector in a matrix notation.

$$\widetilde{\mathsf{A}} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}. \tag{2}$$

An equation written, using boldface letters for vectors will be generally valid w.r.t. to any choice of axes. When there is only system of axes, the three notations can be used, and will be, interchangeably. It is important to distinguish vectors components,$\vec{A}$ $\widetilde{A}$, from the vector $\mathbf{A}$ itself when two or more systems of coordinate axes may be in use.

§3 An Example

Why This Example? To illustrate use of different notations for vectors.

Let K' be a set of axes obtained by carrying out a rotation by and angle α on a set of coordinate axes K. Find relation between components of position vector of a point w.r.t. the two sets K and K'.

Let $\mathbf{R}$ denote the position vector of a point P. The notation for the components along the axes in K and K' will be written as

$$\vec{R} = (x, y, z), \qquad \vec{R}' = (x', y', z') \tag{3}$$

and

$$\widetilde{R} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; \qquad \widetilde{R}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{4}$$

Since the rotation is performed about the Z axis, the Z' axis coincides with Z axis and we have $z' = z$.

However components of $\mathbf{R}$, and also the unit vectors, along X, Y and along X', Y' axes, will be different.

We use the notation $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to denote the unit vectors along K axes. The unit vectors along the K' axes will be denoted by $\mathbf{i}', \mathbf{j}', \mathbf{k}'$, with $\mathbf{k}' = \mathbf{k}$. Thus vector $\mathbf{R}$ can be written in two ways as

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{5}$$

$$\mathbf{R} = x'\mathbf{i}' + y\mathbf{j}' + z\mathbf{k}. \tag{6}$$

Components of $\mathbf{R}$ w.r.t. K' are obtained by taking its dot product with $\mathbf{i}'$ and $\mathbf{j}'$. Thus, from (5) we get

$$x' = \mathbf{R} \cdot \mathbf{i}' = x(\mathbf{i} \cdot \mathbf{i}') + y(\mathbf{j} \cdot \mathbf{i}') \tag{7}$$

$$y' = \mathbf{R} \cdot \mathbf{j}' = x(\mathbf{i} \cdot \mathbf{j}') + y(\mathbf{j} \cdot \mathbf{j}'). \tag{8}$$

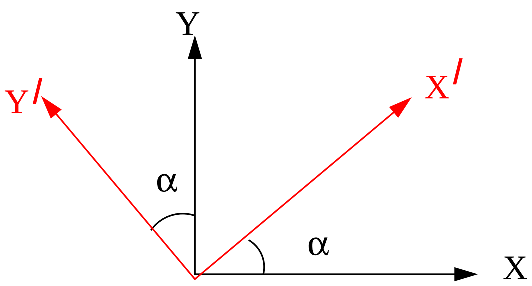

Fig. **1** `Rotation about` Z `axis`

Various dot products can be read in terms of the angle of rotation α from Fig.1 and we get

$$\mathbf{i} \cdot \mathbf{i}' = \cos\alpha, \qquad \mathbf{j} \cdot \mathbf{i}' = \sin\alpha \tag{9}$$

$$\mathbf{i} \cdot \mathbf{j}' = -\sin\alpha, \qquad \mathbf{j} \cdot \mathbf{j}' = \cos\alpha. \tag{10}$$

Substituting the above expressions in Eq.(7)-(8), we get

$$
\begin{aligned}
x' &= x\cos\alpha + y\sin\alpha && (11)\\
y' &= -x\sin\alpha + y\cos\alpha && (12)\\
z' &= z. && (13)
\end{aligned}
$$

and of course $z' = z$. In the matrix notation we writethis relationship as

$$\widetilde{R}' = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \widetilde{R}. \tag{14}$$

§4 Vector Algebra Identities

The dot and cross product satisfy the following identities.

$$
\begin{aligned}
\vec{A}\times(\vec{B}\times\vec{C}) &= (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} && (15)\\
|\vec{A}\times\vec{B}|^2 + (\vec{A}\cdot\vec{B})^2 &= \|A\|^2\|B\|^2 && (16)\\
(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) &= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{B}\cdot\vec{C})(\vec{A}\cdot\vec{D}) && (17)\\
\left[\vec{A}\times\vec{B}, \vec{B}\times\vec{C}, \vec{C}\times\vec{A}\right] &= \left[\vec{A}, \vec{B}, \vec{C}\right]^2 && (18)
\end{aligned}
$$

And some more

$$\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0 \tag{19}$$

$$[\vec{A}, \vec{B}, \vec{C}]\vec{D} = (\vec{A}\cdot\vec{D})(\vec{B}\times\vec{C}) + (\vec{B}\cdot\vec{D})(\vec{C}\times\vec{A}) + (\vec{C}\cdot\vec{D})(\vec{A}\times\vec{B}) \tag{20}$$

If $[\vec{A}, \vec{B}, \vec{C}] \neq 0$, every vector $\vec{X}$ can be represented as

$$\vec{X} = \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C} \tag{21}$$

where

$$\alpha = \frac{\vec{X}\cdot\vec{B}\times\vec{C}}{[\vec{A}, \vec{B}, \vec{C}]}; \quad \beta = \frac{\vec{X}\cdot\vec{C}\times\vec{A}}{[\vec{A}, \vec{B}, \vec{C}]}; \quad \gamma = \frac{\vec{X}\cdot\vec{A}\times\vec{D}}{[\vec{A}, \vec{B}, \vec{C}]} \tag{22}$$

The area of a parallelogram with sides represented by the vectors $\vec{A}, \vec{B}$ is given by $||\vec{A}\times\vec{B}||$ and we have

$$|\vec{A}\times\vec{B}|^2 = \begin{vmatrix} \vec{A}\cdot\vec{A} & \vec{A}\cdot\vec{B} \\ \vec{B}\cdot\vec{A} & \vec{B}\cdot\vec{B} \end{vmatrix} \tag{23}$$

The volume V of a parallelopiped with represented by the vectors $\vec{A}, \vec{B}$ and $\vec{C}$ is given by

$$V^2 = \begin{vmatrix} \vec{A}\cdot\vec{A} & \vec{A}\cdot\vec{B} & \vec{A}\cdot\vec{C} \\ \vec{B}\cdot\vec{A} & \vec{B}\cdot\vec{B} & \vec{B}\cdot\vec{C} \\ \vec{C}\cdot\vec{A} & \vec{C}\cdot\vec{B} & \vec{C}\cdot\vec{C} \end{vmatrix} \tag{24}$$

References Wikipedia , *Vector Algebra Relations*
https://en.wikipedia.org/wiki/Vector_algebra_relations

§5 EndNotes

1. For a quick review of vector algebra see Murphy[1] Ch4; Griffiths[2] Ch1; For use of vectors in Physics see Feynman Lectures Vol-I[3] Ch 11.

2. Watch A Video Watch this video to get started on definition of groups.

References

[1] H. Margenau and G.M. Murphy. *The Mathematics of Physics and Chemistry*. Van Nostrand, 1967.

[2] David J. Griffiths. *Introduction to Electrodynamics*. Pearson Education, Inc, India, 1999.

[3] R. P. Feynman. *Lectures in Physics Vol. I*. B. I. Publishers, INDIA, 1969.