Lecture 8
Radiation from an Accelerated Charge

8.1 Electromagnetic potentials

We have seen (Lienard-Wiechert potentials) that the potentials ϕ and $\mathbf{A}$ at a point $\mathbf{r}$ and time t due to a charge q located at $\mathbf{r}'$ very close to the origin (at the retarded time) are

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(\frac{1}{1-[\mathbf{v}].\mathbf{n}/c}\right) \quad (50)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0 c^2}\frac{1}{r}\left(\frac{[\mathbf{v}]}{1-[\mathbf{v}].\mathbf{n}/c}\right) \quad (51)$$

where $r = |\mathbf{r}-\mathbf{r}'| \approx |\mathbf{r}|$,
$\mathbf{n}$ is the unit vector in the direction of $\mathbf{r}$, and
$[\mathbf{v}]$ is the velocity $\mathbf{v} = d\mathbf{r}'/dt'$ of the particle at the retarded time $t' = t - r/c$.

When the charged particle is moving with a velocity small compared to velocity of light, the term $[\mathbf{v}].\mathbf{n}/c$ can treated as a small quantity (compared to 1), and we can keep it only upto its first order in equations,

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1+\frac{[\mathbf{v}].\mathbf{n}}{c}\right) \quad (52)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0 c^2}\frac{[\mathbf{v}]}{r} \quad (53)$$

In these expressions the dependence on time t comes indirectly through

$$[\mathbf{v}] = \mathbf{v}(t - r/c).$$

Also note that any function of the form

$$F(r,t) \equiv \frac{f(t-r/c)}{r}$$

automatically satisfies the wave equation,

$$\begin{aligned}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)F(r,t) &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)F(r,t) \\ &= 0.\end{aligned}$$

Therefore both ϕ and $\mathbf{A}$ in (52) and (53) satisfy the wave equation far away from the origin $r = 0$.

8.2 Electromagnetic fields

We now calculate electromagnetic fields given by the expressions

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad (54)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (55)$$

In order to get simpler formulas, we take the case of a charged particle moving only along the z-axis near the origin. Then $\mathbf{v} = (0,0,v)$ and $\mathbf{v}.\mathbf{n} = vz/r$.

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} + \frac{q}{4\pi\varepsilon_0}\frac{vz}{r^2 c} \quad (56)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{rc^2}(0,0,v), \quad (57)$$

where it is understood that v is a function of $t - r/c$.

Furthermore, as $1/r$ is small (we call the region where r is large as 'radiation zone') we drop all field terms which are of order $O(1/r^2)$. The leading $O(1/r)$ terms come only from differentiating the velocity $v = v(t - r/c)$. We also omit the common factor $q/4\pi\varepsilon_0$ and only include it in the final expressions. The electric field has two terms

$$\begin{aligned} -\nabla\phi &= O(1/r^2) + \frac{az}{r^3c^2}\mathbf{r} \\ -\frac{\partial \mathbf{A}}{\partial t} &= \left(0, 0, -\frac{a}{rc^2}\right), \end{aligned}$$

where $a = dv/dt'$ is the acceleration of the charge at the retarded time. Similarly, we calculate the magnetic field $B = \nabla\times\mathbf{A}$. Thus the fields in the radiation zone (omitting $O(1/r^2)$ terms) are

$$\mathbf{E}_{\mathrm{rad}} = \frac{q}{4\pi\varepsilon_0}\frac{a}{rc^2}\left(\frac{xz}{r^2}, \frac{yz}{r^2}, \frac{z^2}{r^2} - 1\right) \tag{58}$$

$$\mathbf{B}_{\mathrm{rad}} = \frac{q}{4\pi\varepsilon_0}\frac{a}{rc^3}\left(-\frac{y}{r}, \frac{x}{r}, 0\right). \tag{59}$$

We notice that $\mathbf{E}_{\mathrm{rad}}$ and $\mathbf{B}_{\mathrm{rad}}$ are orthogonal, and perpendicular to the radial vector $\mathbf{r}$. In fact if we use the polar coordinates (r, θ, ϕ) then the $\mathbf{B}_{\mathrm{rad}}$ lines of force are circles of constant θ and r with increasing angle ϕ , whereas $\mathbf{E}_{\mathrm{rad}}$ lines are circles of constant ϕ and r but increasing θ.

The energy radiated in the radial direction $\mathbf{n} = \mathbf{r}/r$ is given by the Poynting vector

$$\mathbf{S}_{\mathrm{rad}} = \varepsilon_0 c^2 \mathbf{E}_{\mathrm{rad}} \times \mathbf{B}_{\mathrm{rad}} = \left(\frac{q^2}{16\pi^2\varepsilon_0}\right)\frac{a^2\sin^2\theta}{r^2c^3}\mathbf{n}. \tag{60}$$

The total energy radiated per unit time is obtained by integrating this expression over the surface of a sphere. That integral is called J. J. Larmor's formula

$$S_{\mathrm{tot}} = \frac{q^2a^2}{6\pi\varepsilon_0 c^3} = \frac{q^2}{4\pi\varepsilon_0}\frac{2}{3}\frac{a^2}{c^3}. \tag{61}$$