Lecture 7
Fields from a Point Charge

The potentials due to a **point charge** were calculated by Lienard (1898) and Wiechert (1900).

Recall (from lecture 3.2) that the potentials in the Lorentz gauge ($\nabla \cdot \mathbf{A} + \partial\phi/(c^2 \partial t)$) satisfy the wave equations,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{r},t) = -\frac{\rho(\mathbf{r},t)}{\epsilon_0} \tag{46}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) = -\frac{\mathbf{j}(\mathbf{r},t)}{\epsilon_0 c^2} \tag{47}$$

The solutions of these using the Green's function for the wave equation are

$$\begin{aligned}
\phi(\mathbf{r},t) &= -\frac{1}{\epsilon_0}\int d^3\mathbf{r}'\int dt' G(\mathbf{r}-\mathbf{r}', t-t')\rho(\mathbf{r}',t') \qquad (48)\\
&= \frac{1}{4\pi\epsilon_0}\int d^3\mathbf{r}'\int dt'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)\rho(\mathbf{r}',t')\\
&= \frac{1}{4\pi\epsilon_0}\int d^3\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/c)
\end{aligned}$$

If the point particle with charge q moves along a trajectory so that at time t the position of the point particle is $\mathbf{s}(t)$ then the charge density is

$$\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{s}(t)). \tag{49}$$

The potential due to this charge is

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\int d^3\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta^3(\mathbf{r}'-\mathbf{s}(t-|\mathbf{r}-\mathbf{r}'|/c))$$

Eveluation of this integral requires some care because the argument of the Dirac delta functions (there are three of them) are not in the variables $\mathbf{r}' = (x^{1'}, x^{2'}, x^{3'})$ but more complicated functions :

$$\delta^3(\mathbf{r}' - \mathbf{s}(t - |\mathbf{r} - \mathbf{r}'|/c)) \equiv \delta^3(\mathbf{f})$$

where

$$\mathbf{f}(\mathbf{r}, \mathbf{r}', t) = \mathbf{r}' - \mathbf{s}(t - |\mathbf{r} - \mathbf{r}'|/c)$$

Actually $\mathbf{r}$ and t are 'sitting' variables doing nothing as far as the integration is concerned. The real functional relationship is between $\mathbf{r}'$ and $\mathbf{f}$. The integration can be changed to variables $\mathbf{f}$ in place of $\mathbf{r}'$

$$\begin{aligned} d^3\mathbf{r}' &= d^3\mathbf{f} \times \frac{1}{J} \\ J &= \det \left\| \frac{\partial f^i}{\partial x'^j} \right\| \end{aligned}$$

The Jacobian matrix can be easily calculated : for example

$$\begin{aligned} \frac{\partial f_i}{\partial x'^j} &= \delta^i_j - \frac{ds^i}{dt} \frac{\partial}{\partial x'^j} \left(\frac{-|\mathbf{r} - \mathbf{r}'|}{c} \right) \\ &= \delta^i_j - \frac{ds^i}{dt} \frac{x^j - x'^j}{c|\mathbf{r} - \mathbf{r}'|} \\ &= \delta^i_j - \frac{v^i n^j}{c} \end{aligned}$$

where $\mathbf{v} = d\mathbf{s}/dt$ is the velocity of the particle and $\mathbf{n} = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ is the unit vector pointing from $\mathbf{r}'$ towards $\mathbf{r}$. You can check that the determinant of this matrix is

$$J = 1 - \mathbf{v} \cdot \mathbf{n}/c$$

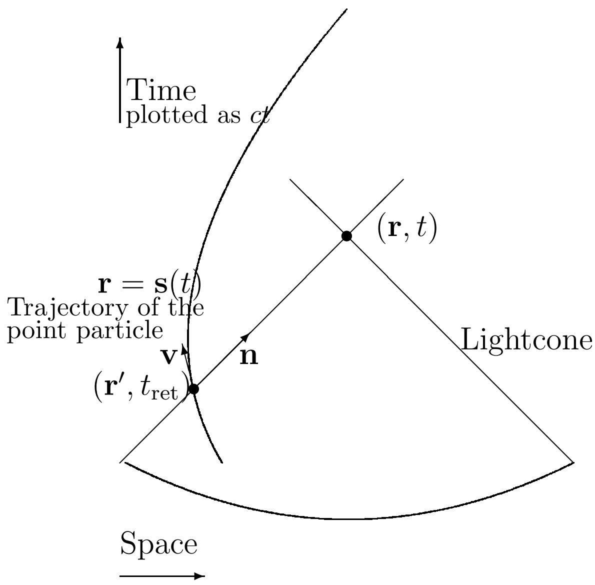
Therefore,

$$\begin{aligned}
\phi(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\int d^3\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta^3(\mathbf{r}'-\mathbf{s}(t-|\mathbf{r}-\mathbf{r}'|/c)) \\
&= \frac{q}{4\pi\epsilon_0}\int d^3\mathbf{f}\delta^3(\mathbf{f})\frac{1}{J}\frac{1}{|\mathbf{r}-\mathbf{r}'(\mathbf{f})|} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{J}\frac{1}{|\mathbf{r}-\mathbf{r}'(\mathbf{f})|}\right]_{\mathbf{f}=\mathbf{0}} \\
&= \frac{q}{4\pi\epsilon_0}\times\frac{1}{1-\mathbf{v}(t_{\rm ret})\cdot\mathbf{n}/c}\times\frac{1}{R}, \qquad R=|\mathbf{r}-\mathbf{s}(t_{\rm ret})|
\end{aligned}$$

Here the solution of

$$\mathbf{f}=\mathbf{r}'-\mathbf{s}(t-|\mathbf{r}-\mathbf{r}'|/c)=0$$

for $\mathbf{r}'$ for a fixed value of $\mathbf{r}$ and t is obtained as follows : Start from the space-time point $\mathbf{r}, t$. Draw the past light cone from this point. Find the space-time point at which the trajectory of the particle $\mathbf{s}(t)$ intersects the past light cone. This point is precisely $\mathbf{r}', t_{\rm ret}$. (See the diagram.) The value of $\mathbf{r}'$ is the position of the charged point particle at $t_{\rm ret}$. So $\mathbf{r}-\mathbf{r}'$ should be replaced by $\mathbf{r}-\mathbf{s}(t_{\rm ret})$ when integration over $\delta^3(\mathbf{f})$ is performed.



You should appreciate the elegance and convenience of the Dirac delta function in getting the Lienard-Wiechert potential by comparing to the original proof which was given in 1900 and is summarized in the book *History of theories of Aether and Electricity* by E. T. Whittaker.