Lecture 6
Fields for Arbitrary Charge and Current Densities

6.1 The wave equations

The Maxwell equations imply wave equations for electric and magnetic fields. Taking 'curl' $\nabla \times (\nabla \times \mathbf{E})$ of the Faraday's law of induction gives

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

where we have interchanges the time and space derivative on the right hand side. Substituting for $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{B}$ from the other Maxwell equations we get

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \frac{1}{\epsilon_0}\nabla\rho + \frac{1}{\epsilon_0 c^2}\frac{\partial \mathbf{j}}{\partial t} \tag{36}$$

Similarly taking 'curl' of $\nabla \times \mathbf{B}$ we can get

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = -\frac{1}{\epsilon_0 c^2}\nabla \times \mathbf{j} \tag{37}$$

These are inhomogeneous wave equations whose particular solution are known to us through the Green's function. The solution of

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{r}, t) = \psi(\mathbf{r}, t) \tag{38}$$

is

$$\phi(\mathbf{r}, t) \;=\; \int d^3\mathbf{r}' \int dt' G(\mathbf{r} - \mathbf{r}', t - t')\psi(\mathbf{r}', t') \tag{39}$$

$$
\begin{aligned}
&= -\int d^3\mathbf{r}' \int dt' \frac{1}{4\pi R}\delta(t-t'-R/c)\psi(\mathbf{r}',t') \\
&= -\int d^3\mathbf{r}' \frac{1}{4\pi R}\psi(\mathbf{r}',t-R/c)
\end{aligned} \tag{40}
$$

where we have substituted the Green's function for the wave equation given by the expression

$$
G(\mathbf{r}-\mathbf{r}',t-t') = -\frac{1}{4\pi R}\delta(t-t'-R/c), \qquad R = |\mathbf{r}-\mathbf{r}'|. \tag{41}
$$

as calculated in lecture 3.4.

Our equations (1) and (2) are of the same from as (3) but they also involve *derivatives*. And handling derivatives requires a little care for the following reason :

The right hand side is an integral over whatever the function ψ is evaluated with t replaced by $t - |\mathbf{r}-\mathbf{r}'|/c$. The function $\psi(\mathbf{r}',t')$ which was a function of one vector variable $\mathbf{r}'$ and one scalar t' respectively has become a function of $\mathbf{r}', \mathbf{r}$ and t. If we had $\partial\psi(\mathbf{r}',t')/\partial x'$, for example, on the right hand side in place of ψ then

$$
\left.\frac{\partial\psi(\mathbf{r}',t')}{\partial x'}\right|_{t'=t-R/c} \tag{42}
$$

would be the quantity inside the integral sign. Which is, bye the way, not the same thing as

$$
\frac{\partial\psi(\mathbf{r}',t-R/c)}{\partial x'} \tag{43}
$$

because of the additional dependence on $\mathbf{r}'$ through R. Another way to say this is that while $\partial/\partial x'$ in (7) means differentiating with respect to x' keeping t' constant, in (8) it means differentiating with respect to x' keeping t and $\mathbf{r}$ constant.

You can ask why we should be bothered to use (8) when (7) is already the correct expression? The answer is that we would like to use one quantity $\psi(\mathbf{r}', t-R/c)$ and its derivatives rather than two quantities $\psi(\mathbf{r}', t-R/c)$ and $[\nabla'\psi(\mathbf{r}', t')]_{t'=t-R/c}$.

The two are related of course.

$$\frac{\partial \psi(\mathbf{r}', t-R/c)}{\partial x'} = \left.\frac{\partial \psi(\mathbf{r}', t')}{\partial x'}\right|_{t'=t-R/c} + \left.\frac{\partial \psi(\mathbf{r}', t')}{\partial t'}\right|_{t'=t-R/c} \frac{\partial}{\partial x'}\left(\frac{-R}{c}\right)$$

Similarly, if we had a time derivative $\partial\psi(\mathbf{r}', t')/\partial t'$, then

$$\left.\frac{\partial \psi(\mathbf{r}', t')}{\partial t'}\right|_{t'=t-R/c}$$

which occurs inside the integral over $d^3\mathbf{r}'$ is related to the derivative w.r.t. t more simply by

$$\frac{\partial \psi(\mathbf{r}', t-R/c)}{\partial t} = \left.\frac{\partial \psi(\mathbf{r}', t')}{\partial t'}\right|_{t'=t-R/c}$$

After this explanation we can proceed to write our solutions.

6.2 Jefimenko's expressions for E and B

For the electric field the right hand side involves $\nabla'\rho(\mathbf{r}', t')$ and $\partial\mathbf{j}(\mathbf{r}', \mathbf{t}')/\partial t'$ both evaluated at $t' = t - R/c$. Using

$$\nabla'\rho(\mathbf{r}', t-R/c) = \nabla'\rho(\mathbf{r}', t')|_{t'=t-R/c} + \left.\frac{\partial \rho(\mathbf{r}', t')}{\partial t'}\right|_{t'=t-R/c} \nabla'\left(\frac{-R}{c}\right)$$

and

$$\nabla' R = \nabla' |\mathbf{r} - \mathbf{r}'| = -\frac{\mathbf{R}}{R} \equiv -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

and an integration by part

$$\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx' \frac{\nabla' f(\mathbf{r}')}{R} = -\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \, \nabla' \left(\frac{1}{R} \right) f(\mathbf{r}')$$

we get (do the algebra!)

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{r}' \Big[\frac{\mathbf{R}}{R^3} \rho(\mathbf{r}', t - R/c) + \frac{\mathbf{R}}{cR^2} \frac{\partial \rho(\mathbf{r}', t - R/c)}{\partial t} \\ &\quad - \frac{1}{c^2 R} \frac{\partial \mathbf{j}(\mathbf{r}', t - R/c)}{\partial t} \Big] \end{aligned} \quad (44)$$

Similarly, the magnetic induction field is obtained as

$$\mathbf{B}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0 c^2} \int d^3\mathbf{r}' \Big[\frac{\mathbf{R}}{R^3} \times \mathbf{j}(\mathbf{r}', t - R/c) + \frac{\mathbf{R}}{cR^2} \times \frac{\partial \mathbf{j}(\mathbf{r}', t - R/c)}{\partial t} \Big] \quad (45)$$

These expressions for **E** and **B** are called Jefimenko's equations.