Lecture 5
Fields for Arbitrary Charge and Current Densities

5.1 Green's function for the Helmholtz equation

The equation to be solved is

$$(\nabla^2 + \kappa^2)G(\mathbf{r}) = \delta(\mathbf{r}),$$

where κ is a *positive* real number. As in the case of Poisson equation we assume

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,\tilde{G}(\mathbf{k})\,e^{i\mathbf{k}.\mathbf{r}}$$

and obtain

$$\tilde{G}(\mathbf{k}) = -\frac{1}{k^2 - \kappa^2}, \qquad k = |\mathbf{k}|.$$

In the Poisson case, $\tilde{G}(\mathbf{k}) = -1/k^2$ gave a problem at $k^2 = 0$ and we redefined the integral for $G(\mathbf{r})$ by changing k^2 to $k^2 + \mu^2$ with $\mu \to 0$. Here, that trick will not do because the singularity cannot be removed that way. (Figure out why?)

In the present case we avoid the singularity at $k = \pm\kappa$ by making κ complex by adding an infinitesimal imaginary part. We can do this in two ways, adding a positive or a negative imaginary part. Both ways are equally valid and give two independent Green's functions.

$$\tilde{G}_{\pm}(\mathbf{k}) = \lim_{\epsilon\to 0} -\frac{1}{k^2 - \kappa_\epsilon^2}, \qquad \kappa_\epsilon = \kappa \pm i\epsilon.$$

The remaining calculation is exactly as that for the Poisson case, (do that), thus

$$G_{\pm}(\mathbf{r}) = -\frac{e^{\pm i\kappa r}}{4\pi r}.$$

5.2 Green's function for the Wave equation

The fundamental equation for the wave equation is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t) = \delta(t)\delta(\mathbf{r}).$$

Traditionally we do it in two steps. First, we define the time-Fourier transform of G

$$G(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{G}(\mathbf{r}, \omega)e^{i\omega t}$$

which shows that the intermediate transform $\hat{G}$ satisfies the Helmholtz equation

$$(\nabla^2 + \kappa^2)\hat{G}(\mathbf{r}, \omega) = \delta(\mathbf{r}), \qquad \kappa = \omega/c$$

because on the right hand side $\delta(t)$ can be written as Fourier transform

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t},$$

and comparing. So, using the solution for the Helmholtz equation, we write

$$\hat{G}_{\pm}(\mathbf{r}, \omega) = -\frac{e^{\pm i\omega r/c}}{4\pi r}.$$

Next, this value of $\hat{G}$ can be substituted to calculate the Green's function

$$\begin{aligned} G_{\pm}(\mathbf{r}, t) &= -\frac{1}{4\pi r}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t \pm r/c)} \\ &= -\frac{1}{4\pi r}\delta\left(t \pm \frac{r}{c}\right). \end{aligned}$$

The two solutions; one with $\delta(t - r/c)$ is called the *retarded* and the one with $\delta(t + r/c)$ is called the *advanced* solution. The reason for these names will appear in the next sections.