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Obtain solution of free particle problem in two dimension using Hamilton Jacobi equation.
Obtain the expression for Hamilton's principal function.

We begin with the Hamiltonian for a free particle in two dimension

$$H = \frac{p^2}{2m} + \frac{p_2^2}{2m} \quad \text{------} \quad (1)$$

The Hamilton's principal function will be written as

$$S = W - Et$$

and can be obtained from Jacobi's complete integral W which obeys the equation

$$\frac{1}{2m}\left(\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right) = E \quad \text{----} \quad (2)$$

where E = energy of the particle. Solve this equation by the method of separation of variables by writing

$$W = W_1(x_1) + W_2(x_2)$$

Then

$$\frac{\partial W}{\partial x_1} = \frac{dW_1}{dx_1}, \quad \frac{\partial W}{\partial x_2} = \frac{\partial W_2}{\partial x_2}. \quad \text{-----} \quad (3)$$

and we get

$$\frac{1}{2m}\left(\frac{dW_1}{dx_1}\right)^2 + \frac{1}{2m}\left(\frac{dW_2}{dx_2}\right)^2 = E \quad \text{----} \quad (4)$$

The first term in (4) depends on x_1 and the second term depends only on x_2. Therefore each term must be a constant. We therefore write

$$\frac{dW_1}{dx_1} = \alpha_1 \quad , \quad \frac{dW_2}{dx_2} = \alpha_2 \qquad \text{- - - (5)}$$

With $\frac{1}{2m}(\alpha_1^2+\alpha_2^2) = E$. - - - - (6)

Integrating (5) we get

$$W_1 = \alpha_1 x_1 + C_1 \quad W_2 = \alpha_2 x_2 + C_2 \qquad \text{- - - - (7)}$$

hence $W(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2 + C_1 + C_2$

Dropping the additive constant C_1+C_2, the complete integral is found to be

$$W(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2.$$

With $H = \left[\left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial x_2}\right)^2\right] / 2m$

$$= \frac{\alpha_1^2+\alpha_2^2}{2m} \qquad (8)$$

We take α_1, α_2 to be the new momenta $\alpha_1, \alpha_2 \to P_1, P_2$

Taking $W(x_1, x_2, \alpha_1, \alpha_2)$ as type II generator the solution to the EOM is given by

$$S = W(x_1\, x_2\, \alpha_1\, \alpha_2) - E(\alpha_1 \alpha_2)\, t \qquad (9)$$

$$\dot{Q}_1 = \frac{\partial H}{\partial \alpha_1} = \alpha_1/m \qquad \dot{Q}_2 = \frac{\partial H}{\partial \alpha_2} = \frac{\alpha_2}{m}$$

$$Q_1 = \frac{\alpha_1 t}{m} + \text{const} \qquad Q_2 = \frac{\alpha_2 t}{m} + \text{const} \qquad (10)$$

from transformation equations

$$Q_1 = \frac{\partial W}{\partial \alpha_1} = x_1 \qquad Q_2 = \frac{\partial W}{\partial \alpha_2}$$

$$p_1 = \frac{\partial W}{\partial x_1} = \alpha_1 \qquad p_2 = \frac{\partial W}{\partial x_2} = \alpha_2 \qquad (11)$$

∴ We get the solution as

$$x_1 = \frac{\alpha_1 t}{m} + x_{10} \qquad p_1 = \alpha_1$$

$$x_2 = \frac{\alpha_2 t}{m} + x_{20} \qquad p_2 = \alpha_2 \qquad (12)$$

where α_1, α_2 are constants.

The Hamilton's principal function is given by (9) expressed in terms of coordinates at t and $t_0 = 0$.

$$\therefore S = \alpha_1 x_1 + \alpha_2 x_2 - \left(\frac{\alpha_1^2 + \alpha_2^2}{2m}\right) t$$

eliminating α_1, α_2 using (12) we get

$$S = x_1 (x_1 - x_{10}) \frac{m}{t} - \frac{m^2}{2m} \frac{(x_1 - x_{10})^2}{t^2} \times t$$

$$+ x_2 (x_2 - x_{20}) \frac{m}{t} - \frac{m^2}{2m} \frac{(x_2 - x_{20})^2}{t^2} \times t$$

$$= \frac{1}{2} m \frac{(x_1 - x_{10})^2}{t} + \frac{1}{2} \frac{m (x_2 - x_{20})^2}{t}$$