☹ me-que-02023

Find rotation matrix for a rotation by an angle α about the axis $1, 2, 1$ where $\cos\alpha = \frac{3}{5}, \sin\alpha = \frac{4}{5}$.

☺ **Solution:** The unit vector along the direction $(1, 2, 1)$ is given by $\hat{n} = \frac{1}{\sqrt{6}}(1, 2, 1)$.
Under a rotation by an angle α about axis $\hat{n} = (n_1, n_2, n_3)$, the new components $\vec{X}$ are related to old components $\vec{x}$ by equation

$$\vec{X} = \vec{x} - \sin\alpha(\hat{n}\times\vec{x}) + (1-\cos\alpha)\hat{n}\times(\hat{n}\times\vec{x})) \tag{1}$$

We compute

$$\hat{n}\times\vec{x} = \big(n_2x_3 - n_3x_2, n_3x_1 - n_1x_3, n_1x_2 - n_2x_1\big)$$

$$\begin{aligned}
(\hat{n}\times(\hat{n}\times\vec{x}))_1 &= n_2(n_1x_2 - n_2x_1) - n_3(n_3x_1 - n_1x_3) && (2)\\
(\hat{n}\times(\hat{n}\times\vec{x}))_2 &= n_3(n_2x_3 - n_3x_2) - n_1(n_1x_2 - n_2x_1) && (3)\\
(\hat{n}\times(\hat{n}\times\vec{x}))_3 &= n_1(n_3x_1 - n_1x_3) - n_2(n_2x_3 - n_3x_2) && (4)
\end{aligned}$$

Therefore

$$\begin{aligned}
X_1 &= (1-\cos\alpha)(n_2(n_1x_2 - n_2x_1) - n_3(n_3x_1 - n_1x_3)) - \sin\alpha(n_2x_3 - n_3x_2) + x_1\\
X_2 &= (1-\cos\alpha)(n_3(n_2x_3 - n_3x_2) - n_1(n_1x_2 - n_2x_1)) - \sin\alpha(n_3x_1 - n_1x_3) + x_2\\
X_3 &= (1-\cos\alpha)(n_1(n_3x_1 - n_1x_3) - n_2(n_2x_3 - n_3x_2)) - \sin\alpha(n_1x_2 - n_2x_1) + x_3
\end{aligned}$$

Therefore

$$\begin{pmatrix} X_1\\ X_2\\ X_3\end{pmatrix} = \mathbb{R}\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}. \tag{5}$$

where the matrix $\mathbb{R}$ is given by

$$\begin{pmatrix}
-(1-\cos\alpha)(n_2^2+n_3^2)+1 & (1-\cos\alpha)n_1n_2 + \sin\alpha n_3 & (1-\cos\alpha)n_3n_1 - \sin\alpha n_2\\
(1-\cos\alpha)n_1n_2 - n_3\sin\alpha & -(1-\cos\alpha)(n_3^2+n_1^2)+1 & (1-\cos\alpha)n_2n_3 + n_1\sin\alpha\\
(1-\cos\alpha)n_1n_3 + n_2\sin\alpha & (1-\cos\alpha)n_2n_3 - n_1\sin\alpha & -(1-\cos\alpha)(n_1^2+n_2^2)+1
\end{pmatrix}$$

Substituting values

$$\hat{n} = \frac{1}{\sqrt{6}}(1, 2, 1), \quad (1-\cos\alpha) = \frac{2}{5}, \quad \sin\alpha = \frac{4}{5}$$

and simplifying gives

$$\begin{pmatrix}
\frac{2}{3} & \frac{2}{15}\left(1+\sqrt{6}\right) & \frac{1}{15}\left(1-4\sqrt{6}\right)\\
\frac{2}{15}\left(1-\sqrt{6}\right) & \frac{13}{15} & \frac{2}{15}\left(1+\sqrt{6}\right)\\
\frac{1}{15}\left(1+4\sqrt{6}\right) & \frac{2}{15}\left(1-\sqrt{6}\right) & \frac{2}{3}
\end{pmatrix} \tag{6}$$