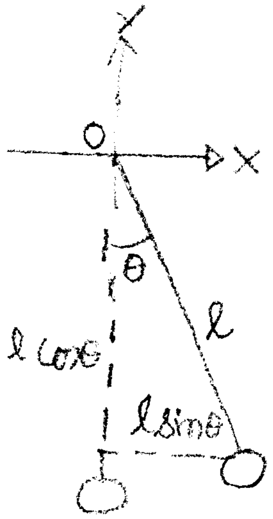
CM007

Determine the period of oscillations of a simple pendulum as a function of amplitude of oscillations.



8.32 Take the point of suspension O as the origin and the coordinate axes as shown. Then the position of the pendulum is

$$x = l\sin\theta \quad y = -l\cos\theta$$

$$K.E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$P.E = mgy = -mgl\cos\theta$$

Total energy

$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

When the amplitude is maximum at $\theta = \theta_0$ we have $\dot{\theta} = 0$

$$\therefore \quad E = -mgl\cos\theta_0$$

Substituting in the energy equation and solving for $\dot{\theta}$ we get

$$-mgl\cos\theta_0 = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$$

or $$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0)$$

$$\therefore \quad d\theta = \quad dt = \sqrt{\frac{l}{2g}}\int\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

If T is the time period, the time taken from $\theta = 0$ to maximum position $\theta = \theta_0$ is $T/4$.

$$\therefore \quad \frac{T}{4} = \int_0^{\theta_0} \frac{d\theta}{\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}}$$

$$\therefore T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{2(\sin^2\frac{\theta_0}{2} - \sin^2\theta/2)}}$$

Let then $\frac{\sin\theta/2}{\sin\theta_0/2} = \sin t$

$$= 4\sqrt{\frac{l}{2g}} \int_0^{\pi/2} \frac{2\,dt\,\sin(\theta_0/2)\cos t}{\cos(\frac{\theta}{2})\sqrt{2\sin^2\frac{\theta_0}{2}(1-\sin^2 t)}}$$

$\frac{\cos\theta/2}{\sin\theta_0/2}\cdot\frac{1}{2}\,d\theta = \cos t\,dt$

$$= 8\sqrt{\frac{l}{4g}} \int_0^{\pi/2} \frac{dt}{\cos\theta/2}$$

$$= 8\sqrt{\frac{l}{4g}} \int_0^{\pi/2} \frac{dt}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2 t}}$$

We expand

$$\frac{1}{\sqrt{1-\sin^2\theta_0/2\,\sin^2 t}} = (1-\sin^2\frac{\theta_0}{2}\sin^2 t)^{-1/2}$$

$$= (1+\frac{1}{2}\sin^2\frac{\theta_0}{2}\cdot\sin^2 t + \dots)$$

$$\int_0^{\pi/2} \frac{dt}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2 t}} = \int_0^{\pi/2} (1+\frac{1}{4}\sin^2\frac{\theta_0}{2}(1-\cos 2t) + \dots)\,dt$$

$$\simeq \frac{\pi}{2} + \frac{1}{4}\frac{\pi}{2}\frac{\theta_0^2}{4} + \dots$$

$$\therefore T = 4\sqrt{\frac{l}{g}}\cdot\frac{\pi}{2}\left(1+\frac{\theta_0^2}{16}+\frac{11}{3072}\theta_0^4 + \dots\right)$$

8:50

12 pts