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$\boxed{\text{CM003}}$

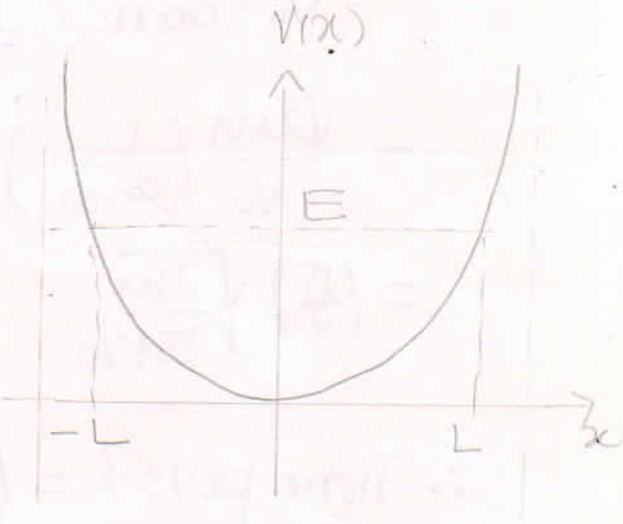
Conservation of energy Solution of EOM in one dimension: Newtonian mechanics

A particle moves in a potential well

$$V(x) = V_0 \tan^2 \alpha x \qquad -\frac{\pi}{2\alpha} < x < \frac{\pi}{2\alpha}$$

Show that the time period of oscillations is

$$T = \left(\frac{\pi}{\alpha}\right)\sqrt{\frac{2m}{E+V_0}}$$



Conservation of energy

$$E = \tfrac{1}{2} m\dot{x}^2 + V(x).$$

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\left(E - V(x)\right)}$$

$$dt = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - V(x)}}$$

Turning points are at $x = \pm L$ where

$$E = V_0 \tan^2 \alpha L.$$

$$T = \sqrt{2m} \int_{-L}^{L} \frac{dx}{\left(V_0 \tan^2 \alpha L - V_0 \tan^2 \alpha x\right)^{1/2}}$$

$$= \sqrt{2m}\,\frac{1}{V_0} \int_{-L}^{L} \frac{dx}{\left(\tan^2 \alpha L - \tan^2 \alpha x\right)^{1/2}} \qquad t = \frac{\tan \alpha x}{\tan \alpha L}$$

$$= \frac{\sqrt{2m}}{V_0} \int_{-L}^{L} \frac{\cos \alpha x \, dx}{\left(\tan^2 \alpha L \cos^2 \alpha x - \sin^2 \alpha x\right)^{1/2}} \qquad dt = \frac{\sec^2 \alpha x}{\tan \alpha L}\,\alpha\, dx.$$

$$\sec^2 \alpha x = 1 + \tan^2 \alpha x$$

$$= \frac{\sqrt{2m}}{V_0} \int_{-L}^{L} \frac{\cos \alpha x \, dx}{\left(\tan^2 \alpha L - (1+\tan^2 \alpha L)\sin^2 \alpha x\right)^{1/2}}$$

$$= \frac{\sqrt{2m}}{V_0} \int_{-L}^{L} \frac{\cos\alpha x \, dx}{(\tan^2\alpha L - \sec^2\alpha L \sin^2\alpha x)^{1/2}}$$

$$= \frac{\sqrt{2m}}{V_0}\left(\frac{1}{\tan\alpha L}\right) \int_{-L}^{L} \frac{\cos\alpha x \, dx}{\left(1 - \frac{\sin^2\alpha x}{\sin^2\alpha L}\right)^{1/2}}$$

$$= \frac{\sqrt{2m}}{V_0}\left(\frac{1}{\cos\alpha L}\right) \int_{-L}^{L} \frac{\cos\alpha x \, dx}{(\sin^2\alpha L - \sin^2\alpha x)^{1/2}}.$$

$$= \frac{\sqrt{2m}}{V_0}\left(\frac{1}{\cos\alpha L}\right) \frac{\pi}{\alpha}$$

$$= \left(\frac{\pi}{\alpha}\right) \sqrt{\frac{2m}{E+V_0}}$$

$$\therefore \text{Time period} = \left(\frac{\pi}{\alpha}\right) \cdot \sqrt{\frac{2m}{E+V_0}}$$

$E = V_0 \tan^2\alpha L$

$= V_0(\sec^2\alpha L - 1)$

$\sec^2\alpha L = \left(\frac{E+V_0}{V_0}\right)$

$\cos\alpha L = \sqrt{\frac{V_0}{E+V_0}}$