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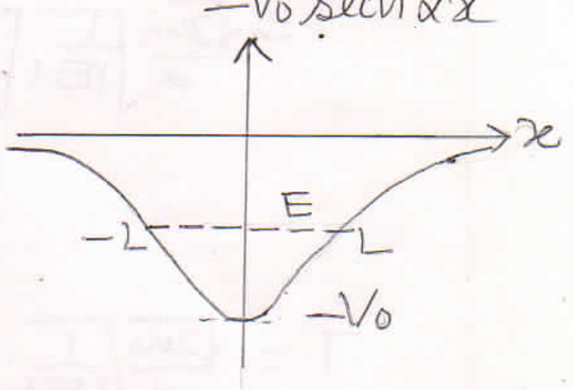
CM002

Keywords: Conservation of energy, solution of equation of motion, one dimension.

For a particle moving in the potential

$$V(x) = -V_0 \operatorname{sech}^2 \alpha x \qquad -V_0 < E < 0$$

find the time period of oscillation if the particle has energy E



Conservation of energy

$$E = \frac{1}{2} m\dot{x}^2 + V(x)$$

$$\dot{x}^2 = \frac{2}{m}(E - V(x))$$

$$dt = \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - V(x)}}$$

Turning pts $x = \pm L$

$$\frac{T}{2} = \int_{-L}^{L} \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E + V_0 \operatorname{sech}^2 \alpha x}} \qquad E = -V_0 \operatorname{sech}^2 \alpha L$$

$$\text{or } |E| = V_0 \operatorname{sech}^2 \alpha L$$

$$\cancel{T = \sqrt{2m} \int_{-L}^{L} \frac{\cosh \alpha x \, dx}{\sqrt{E + V_0 \sinh^2 \alpha x}}}$$

$$= \sqrt{2m} \int_{-L}^{L} \frac{\cosh \alpha x \, dx}{\sqrt{E \cosh^2 \alpha x + V_0}}$$

$$= \sqrt{2m} \int_{-L}^{L} \frac{\cosh \alpha x \, dx}{\sqrt{E + V_0 + E \sinh^2 \alpha x}}$$

$$= \sqrt{2m} \int \frac{1}{\alpha} \frac{dt}{\sqrt{E + V_0 + E t^2}} \qquad \sinh \alpha x = t, \quad \cosh \alpha x \, \alpha \, dx = dt$$

E is negative $E = -|E|$ $\quad -V_0 < E < 0$

$0 < E + V_0$

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$$T = \frac{\sqrt{2m}}{\alpha} \int \frac{dt}{\sqrt{|E|}\sqrt{\frac{E+V_0}{|E|} - t^2}}$$

$$= \frac{\sqrt{2m}}{\alpha}\sqrt{\frac{1}{|E|}} \sin^{-1} t / \frac{E+V_0}{|E|} \Big|_1^2$$

$$= \frac{\sqrt{2m}}{\alpha}\sqrt{\frac{1}{|E|}} \sin^{-1} \frac{\sinh \alpha x}{E+V_0/|E|} \Big|_{x_1}^{x_2}$$

$$= \frac{\sqrt{2m}}{\alpha}\sqrt{\frac{1}{|E|}} \left[\sin^{-1} \frac{\sinh \alpha L}{(E+V_0)/|E|} - \sin^{-1} \frac{\sinh \alpha L}{(E+V_0)/|E|} \right]$$

$$\cosh^2 \alpha L = \frac{V_0}{|E|}$$

$$\sinh^2 \alpha L = \frac{V_0}{|E|} - 1$$

$$= \frac{V_0 - |E|}{|E|}$$

$$T = \frac{\sqrt{2m}}{\alpha}\sqrt{\frac{1}{|E|}} \cdot (\pi/2 + \pi/2)$$

$$= \frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}$$

Time period $= \frac{\pi}{\alpha}\sqrt{\frac{2m}{|E|}}$