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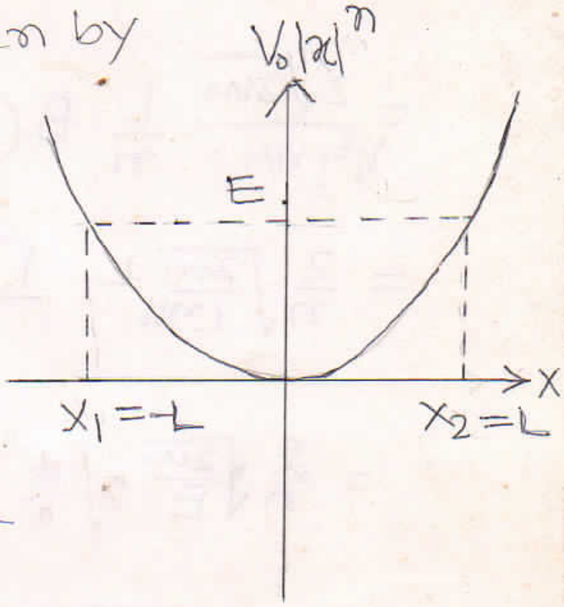
(F)

Key Words Newton's Laws, Solution of EOM
Bounded motion, conservation of energy

Determine the period of oscillation, as a function of energy, when a particle of mass m moves in fields for which the potential energy is given by

$$V(x) = V_0 |x|^n$$



For energy E turning points are given by

$$E = V_0 |x|^n \qquad n \geq 1$$

$$|x| = (E/V_0)^{1/n}$$

$$x_1 = -(E/V_0)^{1/n} \equiv -L \quad x_2 = (E/V_0)^{1/n} \equiv L$$

conservation of energy

$$E = \tfrac{1}{2} m\dot{x}^2 + V(x) \qquad E = V_0 L^n$$

$$\dot{x}^2 = \frac{2}{m}\left(E - V_0 |x|^n\right)$$

Time period = 4 x time taken to go from $x=0$ to $x=L$ (?)

When is this true?

$$dt = \int \frac{dx}{\sqrt{\frac{2}{m}(E - V_0|x|^n)}}$$

$$T = 4\int_0^L \frac{dx}{\sqrt{\frac{2}{m} V_0 (L^n - x^n)}}$$

$$= 4\sqrt{\frac{m}{2V_0}} \frac{1}{L^{n/2}} \int_0^L \frac{dx}{(1-(x/L)^n)^{1/2}}$$

$\frac{x}{L} = t$

$dx = L\,dt$

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$$= 2\sqrt{\frac{2m}{L^n V_0}} \int_0^1 \frac{L\,dt}{(1-t^n)^{1/2}} \qquad t^n = y \quad t = y^{1/n}$$

$$nt^{n-1} = dy \quad dt = \frac{1}{n} y^{1/n - 1} dy$$

$$= \frac{2\sqrt{2m}}{V_0^{1/2} L^{n/2}} \int_0^1 \frac{L}{n} y^{(1-n)/n} (1-y)^{-1/2} dy$$

$$= \frac{2\sqrt{2m}}{V_0^{1/2} L^{n/2}} \frac{L}{n} \cdot B\left(\frac{1-n}{n}+1, \frac{1}{2}\right) \qquad B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

$$= \frac{2}{n}\sqrt{\frac{2m}{L^n V_0}} L \, \frac{\Gamma(1/n)\Gamma(1/2)}{\Gamma(1/n + 1/2)}$$

$$= \frac{2}{n}\sqrt{\frac{2m}{E}} \sqrt{\pi} \left(\frac{E}{V_0}\right)^{1/n} \frac{\Gamma(1/n)}{\Gamma\left(\frac{n+2}{2n}\right)}$$

$$= \frac{2}{n}\sqrt{\frac{2m\pi}{E}} \left(\frac{E}{V_0}\right)^{1/n} \frac{\Gamma(1/n)}{\Gamma\left(\frac{n+2}{2n}\right)}$$

$$\therefore \text{Time period} = \frac{2}{n}\sqrt{\frac{2m\pi}{E}} \left(\frac{E}{V_0}\right)^{1/2} \frac{\Gamma(1/n)}{\Gamma\left(\frac{n+2}{2n}\right)}$$