20.6.2004

CM008

Classical Mechanics

Small oscillations, one dimensions
Newtonian Mechanics

Find the potential if the force acting on a particle in one dimension is

$$F(x) = -m\omega_0^2(x - 2bx^3),\ b>0.$$

Determine the potential energy assuming $V_0 = 0$ and show that the period of oscillations as function amplitude a is

$$\tau = \frac{2}{\omega_0}\int_{-a}^{a} \frac{dx}{(a^2-x^2)(1-b^2(a^2+x^2))^{1/2}}$$

and that for small **b**.

$$\tau \simeq \frac{2\pi}{\omega_0}\left(1 + \frac{3}{4}ba^2\right)$$

10:09 pm

$$F(x) = -m\omega_0^2(x - 2bx^3)$$

$$V(x) = -\int F(x)\,dx + c$$

$$= \frac{1}{2}m\omega_0^2(x^2 - bx^4) + c$$

$$V(0) = 0 \implies c = 0$$

$$\therefore V(x) = \frac{1}{2}m\omega_0^2(x^2 - bx^4)$$

If a is the amplitude of oscillations, then

$$E = V(a) = \frac{1}{2} m\omega_0^2 (a^2 - ba^4)$$

Now

$$E = \frac{1}{2} m \dot{x}^2 + V(x)$$

$$\tau = 2\int_{-a}^{a} \frac{dx}{\dot{x}} = 2\int_{-a}^{a} \frac{dx}{\sqrt{\frac{2}{m} E - V(x)}}$$

$$\therefore \tau = 2\int_{-a}^{a} \frac{dx}{\sqrt{\frac{2}{m}\frac{1}{2} m\omega_0^2 (a^2 - ba^4 - x^2 + bx^4)^{1/2}}}$$

$$= \frac{2}{\omega_0}\int_{-a}^{a} \frac{dx}{\sqrt{(a^2 - x^2) - b(a^4 - x^4)}}$$

$$= \frac{2}{\omega_0}\int_{-a}^{a} \frac{dx}{\sqrt{(a^2 - x^2)(1 - b(a^2 + x^2))}}$$

Change variables $x = a\sin\theta \quad dx = a\cos\theta\, d\theta$

$$\tau = \frac{2}{\omega_0}\int_{-\pi/2}^{\pi/2} \frac{a\cos\theta\, d\theta}{\sqrt{a^2\cos^2\theta\,(1 - ba^2(1 + \sin^2\theta))}}$$

$$= \frac{2}{\omega_0}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{(1 - ba^2 - ba^2\sin^2\theta)^{1/2}}$$

$$= \frac{2}{\omega_0}\frac{1}{\sqrt{1 - ba^2}}\int_{-\pi/2}^{\pi/2} \left\{1 - \frac{ba^2}{1 - ba^2}\sin^2\theta\right\}^{-1/2} d\theta$$

$$\simeq$$

$$\simeq \frac{2}{\omega_0}\frac{1}{\sqrt{1-ba^2}}\int_{-\pi/2}^{\pi/2} d\theta \left\{1+\frac{1}{2}\frac{ba^2}{(1-ba^2)}\sin^2\theta+\cdots\right\}$$

$$= \frac{2}{\omega_0}\frac{1}{\sqrt{1-ba^2}}\left\{\pi+\frac{\pi}{2}\frac{1}{2}\frac{ba^2}{(1-ba^2)}\right\}$$

$$\because \int_{-\pi/2}^{\pi/2}\sin^2\theta\, d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \pi/2$$

$$= \frac{2\pi}{\omega_0}\left(1+\frac{1}{2}ba^2\cdots\right)\left(1+\frac{1}{4}ba^2(1+ba^2+\cdots)\right\}$$

$$\simeq \frac{2\pi}{\omega_0}\left(1+\frac{1}{2}ba^2+\frac{1}{4}ba^2+\cdots\right)$$

$$\text{or}\quad \tau \approx \left(\frac{2\pi}{\omega_0}\right)\left(1+\frac{3}{4}ba^2\right)$$

1.25

$\boxed{10 \text{ pts.}}$