Role of Definitions

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November 9, 2015

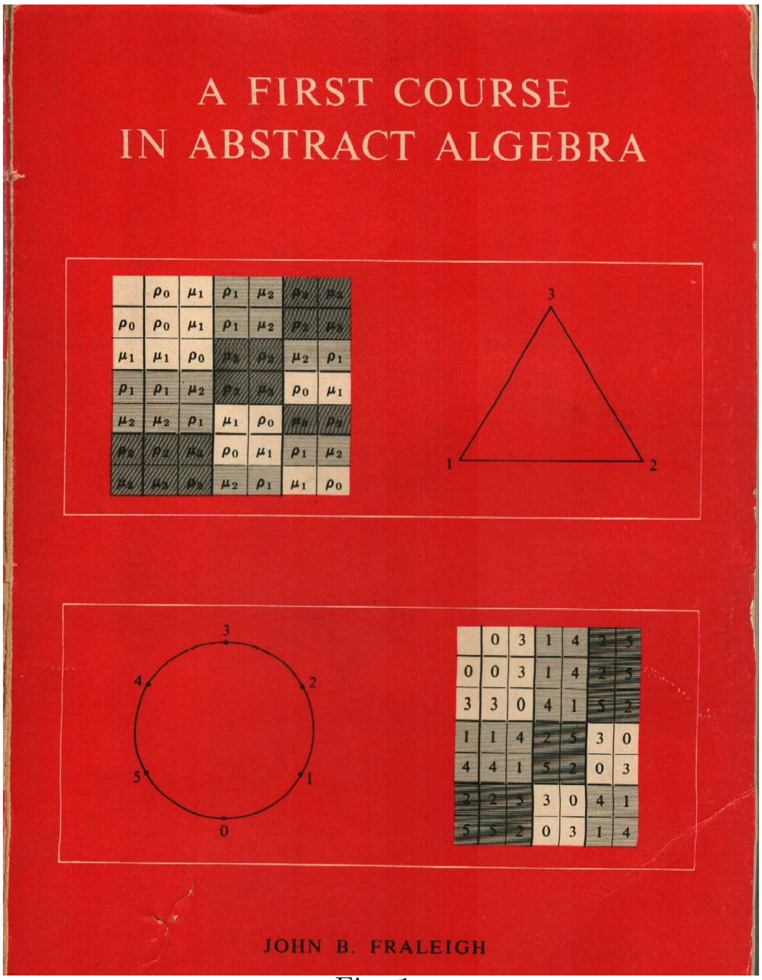

Fig. **1**

The two pages reproduced here appear in Chapter 0 of Fraleigh's Book on Abstract Algebra. This preface is a must for every student of Mathematical Physics and of Mathematics. It tells you importance of definitions. It also emphasises that you need not see proofs of all theorems that you come across, before you start using them.

0 | The Role of Definitions

Many students do not realize the great importance of definitions to mathematics. This importance stems partly from the need for mathematicians to communicate with each other regarding their work. If two people who are trying to communicate on some subject have different ideas as to what is meant by certain technical terms, there can be misunderstanding, friction, and perhaps even bloodshed. Imagine the predicament of a butcher facing an irate housewife who is trying to buy what most people call a rib roast of beef, but who insists on calling it a bottom round roast. The Utopian ideal of complete standardization of terminology unfortunately does not seem to be achievable, even among such precise beings as mathematicians. Quite recently in algebra, for example, some mathematicians have been borrowing the term *division ring* and using it to mean something different from the type of structure which everyone used to call a division ring. *In mathematics, one should strive to avoid ambiguity.*

A very important ingredient of mathematical creativity is the ability to formulate useful definitions, ones which will lead to interesting results. A mathematics student commencing graduate study may find that he spends a great deal of time discussing definitions with fellow graduate students. When the author was in graduate school, a physics graduate student once complained to him that at the evening meal the mathematics students always sat together and argued, and that the subject of their argument was always a definition. A graduate student is usually asked to give several definitions on an oral examination. If he cannot explain the meaning of a term, he probably cannot give sensible answers to questions involving that concept.

Every definition is understood to be an *if and only if* type of statement, even though it is customary to suppress the *only if*. Thus one may define: "A triangle is **isosceles** if it has two sides of equal length," really meaning that a triangle is isosceles if and only if it has two sides of equal length. Now you must not feel that you should memorize a definition word for word. The important thing is to *understand* the concept so that you can define precisely the same concept in your own words. Thus the definition "An **isosceles** triangle is one having two equal sides" is perfectly correct. The definition "An **isosceles** triangle is one having two equal angles" is also correct, for exactly the same triangles are called isosceles in all these definitions.

It is very important for you to note that if some concept has just been defined and you are asked to prove something concerning the concept, you

Fig. **2**

must use the definition as an integral part of the proof. *Immediately after a concept is defined, the definition is the only information one has available regarding the concept.*

This basic importance of definitions to mathematics is also a structural weakness for the reason that not every concept used can be defined. Suppose, for example, one defines the term *set* by "A set is a well-defined collection of objects." One naturally asks what is meant by a *collection*. Perhaps then one defines: "A **collection** is an aggregate of things." What then is an *aggregate*? Now our language is finite, so after some time we will run out of new words to use and have to repeat some words already questioned. The definition is then circular and obviously worthless. Mathematicians realize that there must be some undefined or primitive concept. At the moment they have agreed that *set* shall be such a primitive concept. We shall not define *set*, but shall just hope that when such expressions as "the set of all real numbers," or "the set of all members of the United States Senate" are used, people's various ideas of what is meant are sufficiently similar to make communication feasible.

We summarize briefly some of the things we shall simply assume about sets.

1) A set S is comprised of **elements**, and if a is one of these elements, we shall denote this fact by "$a \in S$".

2) There is exactly one set with no elements. It is the **empty set** and is denoted by "$\varnothing$".

3) A set may be described either by giving a characterizing property of the elements, such as "the set of all members of the United States Senate," or by listing the elements. The standard way to describe a set by listing elements is to enclose the designations of the elements, separated by commas, in braces, e.g., $\{1, 2, 15\}$.

4) A set is **well defined**, meaning that if S is a set and a is some object, then either a is definitely in S, denoted by "$a \in S$", or a is definitely not in S, denoted by "$a \notin S$". Thus one should never say, "Consider the set S of some positive numbers," for it is not definite whether $2 \in S$ or $2 \notin S$. On the other hand, one can consider the set T of all prime positive integers. Every positive integer is definitely either prime or not prime. Thus $5 \in T$ and $14 \notin T$. It may be hard to actually determine whether or not an object is in a set. For example, as this book goes to press it is unknown whether or not $2^{(2^{17})} + 1$ is in T. However, $2^{(2^{17})} + 1$ is certainly either prime or not prime.

It will not be feasible for the student for whom this text is intended to push every definition back to the concept of a set. The author is well aware that he is building on some very naive definitions, especially at the beginning of the text. The first definition we will meet says, "**A binary operation on a set** is a rule . . . set." What on earth is a rule?

Fig. **3**