CM-07 Solved Problem*

Small Oscillations

Three spring problem

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Find normal frequencies, normal modes and normal coordinates of small oscillations for three identical springs system shown in Fig.1 assuming that the system oscillates in a horizontal $X-Y$ plane. The natural length of each spring is $\sqrt{2}$.

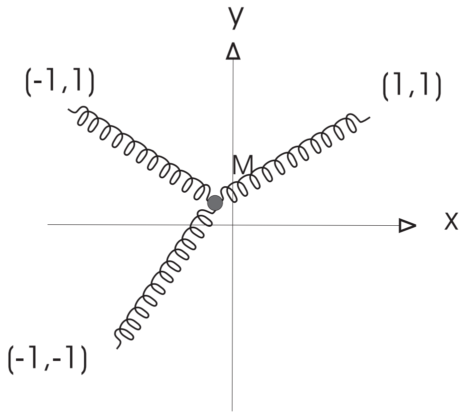

Fig. **1** THREE SPRINGS

☺ **Solution:**

- *Choose generalized coordinates*
  The obvious generalized coordinates to be used are the Cartesian coordinates x, y of the body tied to the springs.

- *Kinetic and energy*
  Now we write the kinetic and potential energies and express them in terms of the generalized coordinates θ, ϕ.

$$\text{K.E.} \quad = \quad \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{1}$$

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- *Potential energy* When the body is at position (x, y), the potential energy of a spring is $\frac{1}{2}k(\Delta L)^2$ where ΔL is the extension or compression of the spring. Therefore we get the potential energies of the three springs as

$$V_1 = \frac{k}{2}\left(\sqrt{(x-1)^2 + (y-1)^2} - \sqrt{2}\right)^2 \tag{2}$$

$$V_2 = \frac{k}{2}\left(\sqrt{(x+1)^2 + (y-1)^2} - \sqrt{2}\right)^2 \tag{3}$$

$$V_3 = \frac{k}{2}\left(\sqrt{(x+1)^2 + (y+1)^2} - \sqrt{2}\right)^2 \tag{4}$$

Hence

$$\text{P.E.} = V_1 + V_2 + V_3 \tag{5}$$

- *Write the Lagrangian* The Lagrangian for the three spring problem becomes given by

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - V_1 - V_2 - V_3 \tag{6}$$

- *Find the equilibrium points and expand the Lagrangian in powers of displacements from equilibrium.* The equilibrium position of the body is obviously the origin, $x = 0, y = 0$. Therefore, we need to expand the potential in powers of x, y and retain terms of up to only second order in x and y. For this purpose we will use the known binomial expansion given by

$$(1+z)^\alpha \approx 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \cdots \tag{7}$$

- Expansion of potential energy terms V_1, V_2, V_3 in powers of x, y.

$$\begin{aligned}
V_1(x,y) &= \frac{1}{2}k\left(\sqrt{(x-1)^2 + (y-1)^2} - \sqrt{2}\right)^2 \\
&= \frac{1}{2}k\left\{x^2 - 2x + 1 + y^2 - 2y + 1 + 2 - 2\sqrt{2}\sqrt{(x-1)^2 + (y-1)^2}\right\} && (8) \\
&= \frac{1}{2}k\{x^2 + y^2 + 2x + 2y + 4 - 2\sqrt{2}(2 - 2x - 2y + x^2 + y^2)^{1/2}\} && (9) \\
&= \frac{1}{2}k\{x^2 + y^2 - 2x - 2y + 4 - 4\left(1 - x - y + \frac{1}{2}(x^2+y^2)\right)^{1/2}\} && (10)
\end{aligned}$$

Do a binomial expansion of $\left(1 + x + y + \frac{1}{2}(x^2+y^2)\right)^{1/2}$ and verify that

$$V_1(x,y) \approx \frac{1}{4}k(x^2+y^2) + \frac{1}{2}kxy \tag{11}$$

In a similar fashion, we would get

$$V_2 \approx \frac{1}{4}k(x^2+y^2) + \frac{1}{2}kxy \tag{12}$$

$$V_3 \approx \frac{1}{4}k(x^2+y^2) - \frac{1}{2}kxy \tag{13}$$

Verify These equations

- *Lagrangian for small oscillations*
  Therefore the Lagrangian for small oscillations takes the form

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \frac{3}{4}(x^2 + y^2) - \frac{1}{2}kxy + \cdots . \tag{14}$$

- *We now write the equations of motion for x and y*

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}, \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{\partial \mathcal{L}}{\partial x}. \tag{15}$$

  The equation of motion take the form

$$m\,\ddot{x} = -k\{\frac{3}{2}x + \frac{1}{2}y\} \tag{16}$$

$$m\,\ddot{y} = -k\{\frac{1}{2}x + \frac{3}{2}y\} \tag{17}$$

$$\tag{18}$$

  Let us use the notation Writing EOM and proceeding this way is simpler for a system with two degrees of freedom. For systems, with more than two degrees of freedom, one should proceed differently.

- It is now helpful to write the EOM in matrix form:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -\nu^2 \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{19}$$

- In a normal mode of vibration all the coordinates oscillate with the same frequency. Therefore we write

$$x = Ae^{i\omega t}, \qquad y = Be^{i\omega t}. \tag{20}$$

  or in matrix form we $\begin{pmatrix} x \\ y \end{pmatrix} = \mathrm{e}^{i\omega t} \begin{pmatrix} A \\ B \end{pmatrix}$Substituting (21) in (19), and canceling $e^{i\omega t}$, we get

$$-\omega^2 \begin{pmatrix} A \\ B \end{pmatrix} = -\nu^2 \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \tag{21}$$

  Rearranging this equation gives

$$\begin{pmatrix} \omega^2 - \frac{3}{2}\nu^2 & -\frac{1}{2}\nu^2 \\ -\frac{1}{2}\nu^2 & \omega^2 - \frac{3}{2}\nu^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \tag{22}$$

  This equation will have nontrivial solution for A, B only if the determinant of the 2×2 matrix on the left is zero.

$$\det \begin{pmatrix} \omega^2 - \frac{3}{2}\nu^2 & -\frac{1}{2}\nu^2 \\ -\frac{1}{2}\nu^2 & \omega^2 - \frac{3}{2}\nu^2 \end{pmatrix} = 0 \tag{23}$$

This determines the frequencies of the normal modes of vibration as solutions of the equation

$$(\omega^2 - \frac{3}{2}\nu^2)^2 - \frac{1}{2}\nu^4 = 0 \tag{24}$$

$$(\omega^2 - 2\nu^2)(\omega^2 - \nu^2) = 0. \tag{25}$$

Hence the two frequencies are given by

$$\omega_1 = \nu = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{2}\nu = \sqrt{\frac{2k}{m}} \tag{26}$$

- *Normal Coordinates*

  Now solve the equations (21), or (22), for A, B for the two frequencies. For the two frequencies, this gives solutions. Write your answers for A, B as column vectors

$$\omega = \omega_1, \qquad \chi_1 = N_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\omega = \omega_2, \qquad \chi_2 = N_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  where N_1, N_2 are some normalization constants.

- We fix normalization constants using

$$\chi_1^{(T)} M \chi_1 = 1 \Longrightarrow 2mN_1^2 = 1 \tag{27}$$

$$\chi_2^{(T)} M \chi_2 = 1 \Longrightarrow 2mN_2^2 = 1 \tag{28}$$

$$\tag{29}$$

- Next we define a matrix S as

$$S = \begin{pmatrix} \chi_1 & \chi_2 \end{pmatrix} = \sqrt{\frac{1}{2m_1}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

- Write the coordinates (x, y) in terms of normal coordinates (Q_1, Q_2) and also the inverse relation

$$\begin{pmatrix} x \\ y \end{pmatrix} = S \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}; \qquad \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = S^{-1} . \begin{pmatrix} x \\ y \end{pmatrix} \tag{30}$$

- Verify that the Lagrangian written in terms of normal coordinates becomes

$$\mathcal{L} = \frac{1}{2}\Big(\dot{Q}_1^2 + \dot{Q}_2^2\Big) - \frac{1}{2}\Big(\omega_1^2 Q_1^2 + \omega_2^2 Q_2^2\Big). \tag{31}$$

✍ `In each normal mode of vibration only one normal coordinate varies harmonically with time, all other normal coordinates remain constant.`

Time variation of coordinates

From Eq.(31), Euler Lagrange equations for the normal coordinates are easy to write and we get

$$\ddot{Q}_1 + \omega_1^2 Q_1 = 0, \qquad \ddot{Q}_2 + \omega_2^2 Q_2 = 0, \tag{32}$$

and the solutions are

$$Q_1(t) = Q_{10}\, e^{i\omega_1 t} + \text{c.c.}, \quad Q_2(t) = Q_{20}\, e^{i\omega_1 t} + \text{c.c.} \tag{33}$$

Here the coordinate amplitudes Q_{10}, Q_{20} are complex numbers and c.c. means complex conjugate of the first term. The time variation of the coordinates (x, y) can now be written down as

$$x(t) = \sqrt{\frac{1}{2m}}(Q_1(t) + Q_2(t)), \quad y(t) = \sqrt{\frac{1}{2m}}(Q_1(t) - Q_2(t)) \tag{34}$$

Taking $Q_{10} = A_1 + iB_1, Q_{20} = A_2 + iB_2$ The four (real) unknown constants A_1, A_2, B_1, B_2, can be determined if initial conditions on position and velocity vectors, (x, y) and $(\dot{x}, \dot{y})$, are given.

✍ In general the time variation of Cartesian coordinates x, y is superposition of both the frequencies. The motion of the body will trace out a Lissajous figure.

✍ Only with special initial conditions, the coordinates will have harmonic time variation with a single frequency.

Questions for you

- Write explicit expressions for most general solution, in terms of the four unknown constants A_1, B_1, A_2, B_2, for the time dependence of normal coordinates. Do the same for the coordinates (x, y).
- Write initial conditions so the the system may vibrate in one of the normal modes. These modes of vibration are called modes of vibration.
- Geometrically describe the motion of the body in each of the two normal modes.

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