Question: Approaching the problem of electric field for a uniformly charged sphere in two different ways, find the value of the integral

$$\vec{\mathcal{E}} = \frac{1}{4\pi\epsilon_0}\iiint d^3r\,'\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}. \tag{1}$$

☺ **Solution:**

Method 1:

A charged sphere has radius R and centre at the origin and having spherically symmetric charge distribution ρ. The electric field for the sphere by at a point *vecr* can be computed by using Gauss law and is given by

$$\vec{E} = \begin{cases} \dfrac{Q(r)}{4\pi\epsilon_0}\vec{r}, & \text{if } r > R,\\ \dfrac{Q(r)}{4\pi\epsilon_0}\dfrac{\vec{r}}{r^3}, & \text{if } r \le R. \end{cases} \tag{2}$$

where $Q(r)$ is the total charge inside a sphere of radius and is

$$Q(r) = \int_0^r \rho(r)\,4\pi r^2\,dr.$$

In case of uniform charge density, $\rho(r)$ independent of r, the above result for the electric field reduces to

$$\vec{E} = \begin{cases} \dfrac{\rho\,\vec{r}}{3\epsilon_0} & \text{if }, r < R,\\ \dfrac{\rho\,\vec{r}}{3\epsilon_0}\times\dfrac{R^3}{r^3}, & \text{if } r < R. \end{cases} \tag{3}$$

On the other hand the Coulomb's law with superposition principle give the electric field as

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\iiint d^3r\,'\rho\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = \frac{\rho}{4\pi\epsilon_0}\iiint d^3r\,'\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \tag{4}$$

for the case of uniformly charged sphere.

Equating the electric field answers from (3) and (4) and canceling ρ we get result

$$\frac{1}{4\pi\epsilon_0}\iiint d^3r\,'\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}. = \begin{cases} \dfrac{\vec{r}}{3\epsilon_0}, & \text{if } r < R,\\ \dfrac{\vec{r}}{3\epsilon_0}\times\dfrac{R^3}{r^3}, & \text{if } r < R. \end{cases} \tag{5}$$

Method 2:
This method makes use of generating function of Legendre polynomials

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_0^\infty t^n P_n(x). \tag{6}$$

We first note that the

$$\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = -\nabla \frac{1}{|\vec{r}-\vec{r}\,'|} \tag{7}$$

Therefore required integral can be rewritten as

$$\iiint d^3r\,' \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = -\nabla \iiint d^3r\,' \frac{1}{|\vec{r}-\vec{r}\,'|^3} \equiv -\nabla f(r) \tag{8}$$

$$\text{where } f(r) = \iiint d^3r\,' \frac{1}{|\vec{r}-\vec{r}\,'|}. \tag{9}$$

and the integral is to be evaluated over a sphere of radius R. because the term with $\vec{r}\,'$in the numerator will vanish when averaged over all directions. Next we consider the two cases when (i) $r > R$ (ii) $r < R$ and use the generating function to expand $1/|\vec{r}-\vec{r}\,'|$ as follows

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_n \left(\frac{r\,'}{r}\right)^n P_n(\cos\theta), \quad \text{if } r > r\,' \tag{10}$$

$$= \frac{1}{r\,'}\sum_n \left(\frac{r}{r\,'}\right)^n P_n(\cos\theta) \quad \text{if } r < r\,' \tag{11}$$

. All terms involving Legendre polynomials $P_n(\cos\theta), n \neq 0$ vanish due to orthogonality property of Legendre polynomials. Thus we get

$$f(r) = \int d\Omega \int_0^R \frac{r\,'^2}{r} = \frac{4\pi R^3}{3r} \tag{12}$$

Substituting the above value in (8) we get the required integral

$$\frac{1}{4\pi\epsilon_0}\iiint d^3r\,' \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = -\frac{1}{4\pi\epsilon_0}\nabla\Big(\frac{4\pi R^3}{3r}\Big) = \frac{1}{3\epsilon_0}\frac{R^3}{r^3}. \tag{13}$$

Case (ii):
In this case also only the first term survives in the expansion in Legendre polynomials. However, the radial expansion needs to be broken up as a sum

of integral over ranges $(0 < r' < r)$ and $(r < r' < R)$. Using appropriate expression from (10) and (11), we get

$$f(r) = 4\pi\Big[\int_0^r (1/r)r'^2 dr' + \int_r^R (1/r')r'^2 dr'\Big] \quad (14)$$

$$= 4\pi\Big[\frac{r^2}{3} + \frac{R^2 - r^2}{2}\Big] = (4\pi)\Big(\frac{R^2}{2} - \frac{r^2}{6}\Big) \quad (15)$$

Taking gradient of $f(r)$, (8) gives the final answer as

$$\frac{1}{4\pi\epsilon_0}\iiint d^3r'\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{\vec{r}}{3\epsilon_0}. \quad (16)$$