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To find α so that the trial wave function

$$u(x) = N \exp(-\alpha^2 x^2/2)$$

may give best estimate for ground state energy of a particle in Dirac δ function potential.

$$V(x) = -g\delta(x) \qquad g>0$$

First fix the normalization

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = 1$$

or $\int_{-\infty}^{\infty} |N|^2 \exp(-\alpha^2 x^2)\, dx = 1 \Rightarrow N^2 = \frac{\alpha}{\sqrt{\pi}}$

Next compute average of kinetic energy

$$\langle KE \rangle = \int_{-\infty}^{\infty} u^*(x) \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right) u(x)\, dx$$

$$= -\frac{\hbar^2}{2m} N^2 \int_{-\infty}^{\infty} (-\alpha^2 + \alpha^4 x^2)\, e^{-\alpha^2 x^2} dx$$

$$= \cdots = \frac{\hbar^2}{4m\alpha^2}$$

$$\langle P.E. \rangle = \int_{-\infty}^{\infty} u^*(x) (-g\delta(x))\, u(x)\, dx$$

$$= -g|u(0)|^2 = -g|N|^2 = -\frac{g\alpha}{\sqrt{\pi}}$$

$$\langle E \rangle_\alpha = \frac{\hbar^2 \alpha^2}{4m} - \frac{g\alpha}{\sqrt{\pi}}$$

minimize energy w.r.t. α

$$\frac{\partial E}{\partial \alpha} = 0 \Rightarrow \left(\alpha = \frac{2gm}{\hbar^2\sqrt{\pi}}\right)$$

$$\therefore \alpha = \frac{2gm}{\hbar^2\sqrt{\pi}}$$

Note: Useful integrals to remember

$$\int_{-\infty}^{\infty} \exp(-\lambda x^2)\, dx = \sqrt{\frac{\pi}{\lambda}}$$

Differentiate w.r.t. λ to get

$$-\int_{-\infty}^{\infty} x^2 \exp(-\lambda x^2)\, dx = -\frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}}$$

$$\therefore \int_{-\infty}^{\infty} x^2 \exp(-\lambda x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}}$$

Remark Corresponding energy value turns out to be

$$E = \frac{\hbar^2\alpha^2}{4m} - \frac{g\alpha}{\sqrt{\pi}} = \frac{\hbar^2}{4m}\left(\frac{4g^2m^2}{\hbar^4\pi}\right) - \frac{g}{\sqrt{\pi}}\left(\frac{2gm}{\hbar^2\sqrt{\pi}}\right)$$

$$= \frac{-g^2m}{\pi\hbar^2}$$

Which is greater than the exact value $-\frac{mg^2}{2\hbar^2}$, as it should be.