Electrodynamics

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Lecture 3.1 : Units and Dimensions

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§ 1

The SI system of units

You already know about the mechanical units of kilogram, meter and second representing the dimensions of mass M, length L and time T. In terms of these units the unit of energy is a **joule**, unit of force is a **newton** and so on.

It is traditional in physics to name units after great physicists who contributed to knowledge closest to the area where that unit is employed. As unit names these are written without the capital letter, with no disrespect intended to the great men. And women. (What and how much is one curie?)

Electromagnetic fields require an additional dimension.

§ 2

Charge and Current

The natural choice for the additional dimension should be **charge** because the study of electromagnetic phenomena started with charge . However the fundamental electromagnetic quantity in the SI system is taken to be the electric **current**. Electric current i is defined as the flow of charge per unit time in a conducting wire. The unit of current is called an **ampere** equal to one **coulomb** per second where a coulomb is the unit of charge. The physical dimension of current will be denoted by [current]. (In general, the dimension of a quantity will be represented by that quantity enclosed by square brackets. We will be a little sloppy and use the same symbol for the units of the quantity as well.)

The unit of charge, **coulomb** has dimensions [charge] = [current]×[time].

It is a fact of nature, and it is something we do not fully understand, that the electric charge is always found to be a multiple of some fundamental natural unit of charge. This unit is taken by convention to be the charge of the electron. In this convention a coulomb is such that the electron charge is -1.6×10^{-19} coulombs.

(Add comments on fractional quark charges, Dirac charge quantization condition and running coupling constant in quantum electrodynamics.)

§ 3

Electric Field E and Displacement D

Electric field E is the force per unit charge and has dimensions

$$[\mathbf{E}] = \frac{[\text{force}]}{[\text{charge}]} = \frac{\text{newton}}{\text{ampere} \times \text{second}} \tag{1}$$

Another electrical unit is volt. A charge q gains energy $qE\,dx$ under the electric field of magnitude E when it moves a distance dx in the same direction as the field. The energy Edx gained per unit charge is called electric potential, and it is measured in a unit **volt**. Therefore, a useful unit for E is

$$[\mathbf{E}] = \frac{\text{volt}}{\text{meter}} \tag{2}$$

There is a quantity **D** which represents the influence or excitation that an electric field has on matter. Maxwell called it **dielectric displacement** and the name refers to the polarization or separation of positive and negative charges of neutral matter under the influence of **E**. The atomic nature of matter was still unclear in the second half of the nineteenth century. The free charges were supposed to cause the electric fields **E** which in turn caused the displacement **D**. The relation was given by the Gauss' law

$$\int_V \mathbf{D} \cdot \mathbf{n} \, da = \text{sum of free charges inside } V$$

This determines

$$[\mathbf{D}] = \frac{[\text{charge}]}{[\text{area}]} = \frac{\text{coulomb}}{\text{meter}^2} \tag{3}$$

This is a quantity very different from **E**. But it was found that the two are proportional to each other in most situations $\mathbf{D} = \epsilon \mathbf{E}$. When there is no matter but only free charges, the constant ϵ takes the value ϵ_0, called the **permittivity of empty space**

$$\mathbf{D} = \epsilon_0 \mathbf{E}. \tag{4}$$

It which has the dimensions

$$[\epsilon_0] = \frac{[\mathbf{D}]}{[\mathbf{E}]} = \frac{\text{coulomb}}{\text{meter}^2} \times \frac{\text{meter}}{\text{volt}} = \frac{\text{farad}}{\text{meter}} \tag{5}$$

Here we use the equation $Q = CV$ relating the voltage and charge of a capacitor and employ **farad** (equal to one coulomb per volt) as the unit of capacity. The numerical value of ϵ_0 is given by the relation

$$\epsilon_0 = \frac{10^7}{4\pi \times 9 \times 10^{16}} \approx 8.846 \times 10^{-12} \frac{\text{farad}}{\text{meter}} \tag{6}$$

§ 4

The Electromagnetic Constant c

There is another constant c of electromagnetic theory which has dimensions of velocity. Before the SI system of units was standardized, there were two principal systems - the so-called **electrostatic units** or esu and **electromagnetic units** or emu. In order to avoid a new dimension for electromagnetic quantities, it was decided to call an electrostatic unit of charge as that charge which repels by unit force an equal amount of charge at a unit distance by Coulomb's law.

$$[\text{force}] = \frac{[\text{charge}]^2}{[\text{distance}]^2} \tag{7}$$

This gives the dimensions of esu of charge in terms of mechanical dimensions

$$[\text{charge}]_{\text{esu}} = \sqrt{[\text{force}]}[\text{distance}] \tag{8}$$

On the other hand the electromagnetic definition depends on Ampere's law about attraction between current carrying wires. Recall that the force between two parallel 'current elements' at the ends of a line perpendicular to them is proportional to the product $(i_1 dl_1)(i_2 dl_2)$ and inversely proportional to square of distance between them. An emu of current is then the amount of current i which produces a force equal to $(dl)^2$ when equal and parallel current elements idl are kept a unit distance apart. Then

$$[\text{force}] = \frac{[\text{current}]^2_{\text{emu}}[\text{distance}]^2}{[\text{distance}]^2} = [\text{current}]^2_{\text{emu}} \tag{9}$$

Therefore,

$$[\text{current}]_{\text{emu}} = \sqrt{[\text{force}]} \tag{10}$$

An emu of charge then, is defined as the charge which passes through a wire with one emu of current in one second. It has the dimension

$$[\text{charge}]_{\text{emu}} = \sqrt{[\text{force}]}[\text{time}] \tag{11}$$

There is no reason for an esu of charge to match with an emu of charge. These quantities do not even have the same dimension! Actually, dimension-wise

$$\frac{[\text{charge}]_{\text{esu}}}{[\text{charge}]_{\text{emu}}} = [\text{velocity}] \tag{12}$$

which means that whenever charge is expressed in electrostatic units, we can convert it into electromagnetic units of charge by multiplying by a constant of the dimension of velocity.

This constant was called c and it was measured rather carefully by Weber and Kohlrausch in 1856. Their result was $c = 3.1 \times 10^{10}$ cm/sec. The extreme closeness of c to speed of light in vacuum made Kirchoff suggest a theory where the speed of propagation of waves of electric disturbances in a perfect conductor was equal to this constant c. Five years later in 1861 Maxwell gave his theory of propagation of electromagnetic waves, and it was recognized that light must be the same phenomenon. Electromagnetic waves were produced in the laboratory by Heinrich Hertz around 1888 and the measured speed agreed with the speed of light. Note, incidentally, that c is a large number (in usual units). Therefore **the emu of charge must be a very large unit compared to esu of charge**. Roughly speaking, magnetic effects are weaker by a factor of c compared to electric effects.

The the speed of light in vacuum in recent measurements is :

$$c = 2.997924590(8) \times 10^{8} \frac{\text{meter}}{\text{second}} \tag{13}$$

§ 5

Magnetic Induction B and Magnetic Intensity H

Let us come to the units of magnetic type. Magnetic fields were discovered through natural magnets, and it was found to that they always occurred as magnetic moments or magnetic dipoles. The force between these magnetic dipoles was similar to the force between electric dipoles, and could be derived by assuming a similar inverse square law between poles. If we call by magnetic **pole-strength** the quantity analogous to the electric charge, then the natural quantity to define is a magnetic field (similar to the electric field)

$$[\mathbf{B}] = \text{“the magnetic field”} = \frac{[\text{force}]}{[\text{pole strength}]} \tag{14}$$

Unfortunately, due to historical confusions and careless nomenclature, the currently used word for **B**, the rightful owner of the name magnetic field, is **magnetic induction**.

Adding insult to injury, there is a totally different quantity **H** which is given the name **magnetic field intensity**. We shall come to that shortly.

It was discovered by Ampere that a small closed wire loop of area A and with current i behaves exactly as if it was a magnet of dipole magnitude iA. This fact can be used to relate the quantity pole-strength to current :

$$[iA] = \text{ampere} \times \text{meter}^2 = [\text{pole strength}] \times \text{meter} \tag{15}$$

or

$$[\text{pole strength}] = \text{ampere} \times \text{meter} \tag{16}$$

Thus the unit of **B** turns out to be

$$[\mathbf{B}] = \frac{\text{newton}}{\text{ampere} \times \text{meter}} \tag{17}$$

Notice the relation between dimensions of **E** and **B** :

$$[\mathbf{B}] = \frac{[\text{force}]}{[\text{charge/time}][\text{length}]} = \frac{[\mathbf{E}]}{[\text{velocity}]} \tag{18}$$

The unit of magnetic induction is **tesla**. Sometimes an older, smaller, unit called **gauss** is also used. One tesla is equal to 10^4 gauss.

The electric displacement **D** includes a term the effects of electric polarization which is measured by electric dipole moment per unit volume with the dimensions [charge][length]/[volume] or [charge]/[area].

Similarly, there is a quantity **H** called the the **magnetic field intensity**. **H** represents the magnetic excitation in a material due to magnetic induction **B**. It includes magnetic polarization which has the dimensions of magnetic moment per unit volume or (what is the same thing), [pole-strength]/[area] in complete analogy to the electric case. Thus,

$$[\mathbf{H}] = \frac{[\text{pole-strength}]}{[\text{area}]} = \frac{\text{ampere}}{\text{meter}} \tag{19}$$

Just as **D** and **E** are related linearly, so are **H** and **B** are related linearly for fields not too strong. But the proportionality constant μ is written *not* as $\mathbf{H} = \mu\mathbf{B}$ as one would expect because **H** is dependent on **B** but as $\mathbf{B} = \mu\mathbf{H}$ ("which is illogical" grumbled the great physicist Arnold Sommerfeld.)

Be that as it may, the constant μ when there is no magnetic matter around, is written μ_0 which is called the **magnetic permeability** of empty space. It has the dimensions

$$[\mu_0] = \frac{[\mathbf{B}]}{[\mathbf{H}]} = \frac{\text{newton}}{\text{ampere}^2} \tag{20}$$

"There is one more horrible thing.", R. P. Feynman says. When magnetic materials *are* present, (and μ is different form

μ_0) the quantity $\mu_0\mathbf{H}$ which has the same dimensions as $\mathbf{B}$ is measured in terms of a unit called 'oersted'. One oesrted is exactly equal to one gauss, they have the same dimension, but sadly, conventions have no logic. Luckily, the use of oersted is limited only to old books.

You can check that the dimensions of μ_0 and ϵ_0 are related by a constant of the dimensions of a square of velocity

$$[\mu_0] = \frac{1}{[\epsilon_0][\text{velocity}]^2}.$$

But there are no prizes for guessing what this velocity constant is. As a matter of fact,

$$\epsilon_0\mu_0 = \frac{1}{c^2} \tag{21}$$

Lecture 3.2 : Maxwell's Equations

Was it a God who wrote these lines ...?

Ludwig Boltzmann

(*Vorlesungen uber Maxwells Theorie der Electrizitat und des Lichtes*,Vol II, Munchen, 1893)

§ 1

<u>The Basic Fields E and B</u>

Electric and magnetic fields are two facets of the same field called electromagnetic field (em-field for short). The em-field is created by charge and current densities (ρ and $\mathbf{j}$ respectively). They are governed by **Maxwell's equations** :

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \text{(Gauss' Law)} \tag{22}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{23}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(Faraday's Law)} \tag{24}$$

$$c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}. \tag{25}$$

These equations are supposed to determine $\mathbf{E}$ and $\mathbf{B}$ which are six quantities dependent on spacetime. These are eight equations (two scalar and two vector equations). Are these over-determined?

§ 2

Conservation of charge

Surely, the maxwell's equations do not hold for any arbitrary independent values of ρ and $\mathbf{j}$. Taking the dot product with ∇ in the last of the Maxwell equations the left hand side is zero because $\nabla.(\nabla \times \text{(any vector)}) = 0$ and the right hand side is

$$\frac{1}{\epsilon_0}\nabla \cdot \mathbf{j} + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{E})$$

If we substitute from the first of the equations we get the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{26}$$

which simply tells us that in a volume V with its surface S the depletion of charge per unit tine is accounted for by flow of charge from the surface :

$$\int_S \mathbf{j}.\hat{\mathbf{n}} da = \int_V \nabla \cdot \mathbf{j} dv = -\frac{\partial}{\partial t}\int_V \rho dv$$

§ 3

Poincare Lemmas

We know that $\mathbf{A}$ is any vector field then

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0.$$

What is intersting is that, the converse is also true, at least in a 'small region'. If we find that for some vector field $\mathbf{B}$

$$\nabla.\mathbf{B} = 0$$

then $\mathbf{B}$ must be of the form $\nabla \times \mathbf{A}$ in a neighborhood.

Similarly, we know that $\nabla \times (\nabla\psi)$ is always zero. The Poincare lemma for this case says that if for some vector field $\mathbf{C}$ it is true that $\nabla \times \mathbf{C} = 0$ then in a small region there exists a scalar function ϕ such that we can express $\mathbf{C} = \nabla\phi$.

These lemmas are very useful. The 'small region' or neighborhood is very often the whole space in simple cases which we deal with. The exceptions will be discussed separately.

§ 4

Potentials

Two of the Maxwell equations do not involve any charge or current. They are

$$\nabla \cdot \mathbf{B} = 0, \quad \text{and} \quad \nabla \times \mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t}$$

Using the Poincare lemmas we can first write

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

This vector field is called **vector potential** or sometimes *magnetic* vector potential. Once $\mathbf{B}$ is so determined, we can put it in the Faraday's law

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \qquad (27)$$

where the Poincare lemma can be used again and we write

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi.$$

The field ϕ is called the **scalar potential**. The negative sign is traditional and has the same origin as the mechanical equation 'force = - gradient of potential'.

§ 5

Gauge freedom

Electromagnetic fields do not determine the potentials completely. If magnetic induction $\mathbf{B}$ is given to us then vector potential $\mathbf{A}$ and $\mathbf{A}' \equiv \mathbf{A} + \nabla\Psi$ give the same field $\mathbf{B}$ because $\nabla \times \nabla\Psi = 0$. The electric field is

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi - \frac{\partial \mathbf{A}'}{\partial t} + \frac{\partial}{\partial t}\nabla\Psi$$

therfore if we choose a new scalar potential $\phi' = \phi - \partial\Psi/\partial t$ then even the electric field does not change :

$$\mathbf{E} = -\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t}$$

The transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\Psi, \qquad \phi \to \phi - \partial\Psi/\partial t$$

is called a **gauge transformation** and the scalar function Ψ which governs the gauge transformation as the **gauge function**.

§ 6

Gauge Fixing

It is simpler to work with potentials rather than em-fields $\mathbf{E}$ and $\mathbf{B}$ because in place of the two vector fields we can deal with only one vector field $\mathbf{A}$ and one scalar field ϕ. Moreover, we can forget about two of the Maxwell equations because they are automatically satisfied.

In terms of the potentials, the remaining Maxwell equations look like

$$\begin{aligned}\nabla^2\phi + \frac{\partial}{\partial t}\nabla\cdot\mathbf{A} &= -\frac{\rho}{\epsilon_0}\\ c^2\nabla(\nabla\cdot\mathbf{A}) - c^2\nabla^2\mathbf{A} &= \frac{\mathbf{j}}{\epsilon_0} - \nabla\left(\frac{\partial\phi}{\partial t}\right) - \frac{\partial^2\mathbf{A}}{\partial t^2}\end{aligned}$$

which can be rearranged as

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} - \frac{\partial}{\partial t}\nabla\cdot\mathbf{A}, \tag{28}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\frac{\mathbf{j}}{\epsilon_0 c^2} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right). \tag{29}$$

The non-uniqueness of potentials is not a big problem. We can restrict the freedom in the choice of potentials by imposing restrictions of our own on the potentials. Such restrictions may 'fix' the potentials wholly or partly. We can also choose the condition on potentials conveniently to simplify our equations.

Out of the infinitely many ways of gauge fixing there are two popular choices.

1. We impose

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0. \tag{30}$$

   This choice is called the **Lorentz gauge** and the Maxwell equations become

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -\frac{\rho}{\epsilon_0}, \tag{31}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\frac{\mathbf{j}}{\epsilon_0 c^2}. \tag{32}$$

2. We require

$$\nabla \cdot \mathbf{A} = 0, \tag{33}$$

and this choice is called the **Coulomb gauge**. In this case the Maxwell equations look like

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{34}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\frac{\mathbf{j}}{\epsilon_0 c^2} + \frac{1}{c^2}\nabla\frac{\partial \phi}{\partial t} \tag{35}$$

Lecture 3.3
Generalized Functions

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§ 7

Generalised functions

The standard definition of a function f is as a mapping from the set $\mathcal{R}$ of real numbers (or a suitable interval of it) into the real numbers, so that it assigns a number $f(x)$, called its value, to $x \in \mathcal{R}$.

Consider the "step function" defined as follows :

$$\begin{aligned} \theta(x) &= 0 \quad x < 0 \\ \theta(x) &= 1 \quad x > 0 \end{aligned}$$

This is very much like an ordinary function, in fact a constant function, everywhere except the point $x = 0$ where it is not defined. The function is discontinuous at $x = 0$ and we can not define its derivative at that point.

The theory of generalised functions is a genralization of the concept of functions to include functions which may have discontinuities or singularities at some or other point of their domain of definition.

For this purpose we must look at an alternative way to define a function.

There are three different ways to define generalized functions.

1. A generalized function is defined by a sequence of ordinary functions which "tend towards" the singular function.

2. A generalized function defined indirectly when integral of its product with a smooth well behaved functions is given.

3. A generalized function is defined as boundary value of an analytic function.

All three methods are used and they complement each other.

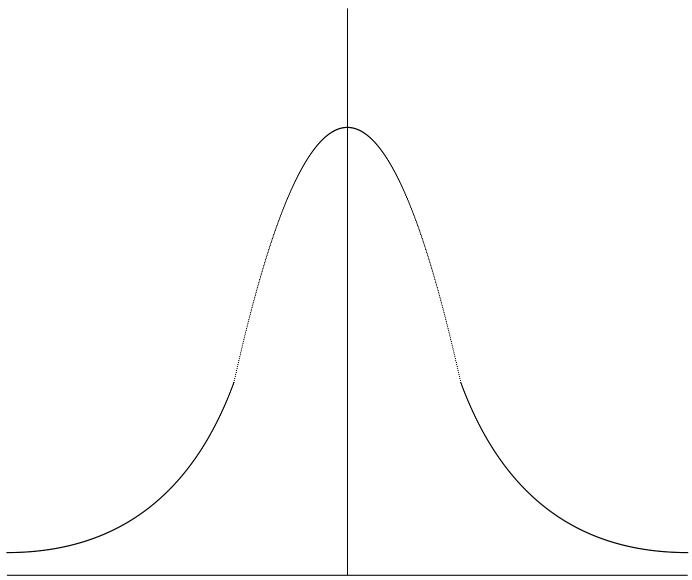
§ 8

Sequence of functions

The best example is the Dirac delta function. The sequence of functions is chosen as

$$f_n(x) = \sqrt{\frac{n}{\pi}} e^{-nx^2}, \qquad n = 1, 2, \ldots$$

The functions looks like



The "area under the curve" of f_n is $\int_{-\infty}^{\infty} f_n(x)dx = 1$ (check that). And for larger values of n the functions become narrowly and sharply peaked around $x = 0$ always keeping the area under the curve equal to 1.

The Dirac delta function is the limiting function of this sequence.

This is not the only sequence of functions which defines the Dirac delta. There are several (in fact infinitely many) such sequences. Another sequence of functions is obtained by

$$\frac{1}{\pi}\frac{\epsilon}{x^2 + \epsilon^2}$$

as $\epsilon \to 0$. If you really must insist on a sequence, you can take $\epsilon = 1/n$ which is equivalent to $\epsilon \to 0$ as $n \to \infty$. This gives

$$g_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}, \qquad n = 1, 2, \ldots$$

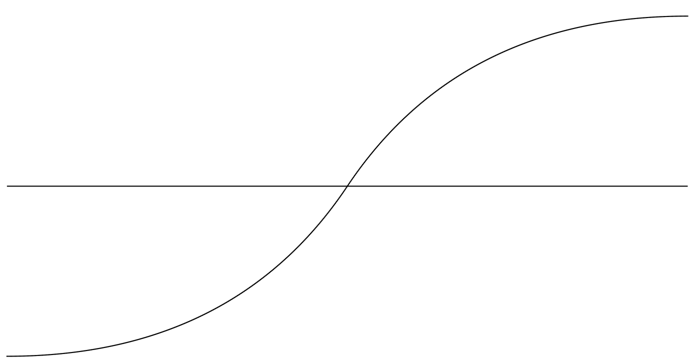
These functions also have unit area under the curve and for large values of n the functions become very sharply peaked and narrow near $x = 0$.

Actually it does not matter which particular sequence is used for the definition.

The step function can be approximated by a sequence of functions

$$h_n(x) = \frac{1}{2} + \frac{1}{2}\tanh(nx)$$

The function $\tanh(nx)$ looks like



For large values of n the function becomes more and more steep at origin and for most of the positive side it is practically equal to 1 and on the negative side it is -1. Another sequence is

$$k_n(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(nx) \qquad n = 1, 2, \ldots$$

where it is understood that we take the values of $\tan^{-1}(x)$ in the range $-\pi/2$ to $\pi/2$. The function $\tan^{-1}(x)$ also has graph like tanh(x), and suitable normalization has been done to secure the right values in the limit to give the step function.

A third interesting example is the "Cauchy principal value" of $1/x$, This function is obtained by omitting the singular part of $1/x$ in a symmetrical way from the neighbourhood of $x = 0$. Let ϵ be a small number, then we define the Caucy principal value denoted by

$$P\left(\frac{1}{x}\right)$$

as the limit $\epsilon \to 0$ of the function

$$\begin{aligned} P\left(\frac{1}{x}\right) &= \frac{1}{x} \qquad (|x| > \epsilon) \\ &= 0 \qquad (|x| < \epsilon) \end{aligned}$$

Again we see the discontinuities, We can define it by a sequence of functions

$$C_n(x) = \frac{x}{x^2 + \epsilon^2} = \frac{n^2 x}{1 + n^2 x^2} \qquad (\epsilon = \frac{1}{n})$$

You must plot these functions. The idea is that for $|x| >> \epsilon$ the function behaves like $1/x$ and near $x = 0$ it is linear with a large slope (n^2). The turning point from $1/x$ to x behavior is at $x = \epsilon = 1/n$.

We are already in a position to prove an important relation :

$$\lim_{\epsilon \to 0} \frac{1}{x \pm i\epsilon} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

The left hand side is a a complex function with a small imaginary part

$$\frac{1}{x \pm i\epsilon} = \frac{x}{x^2 + \epsilon^2} \mp i\frac{\epsilon}{x^2 + \epsilon^2}$$

When $\epsilon \to 0$ the first term on the right hand side becomes the Cauchy principal value, and the second term gives Dirac delta function. Therefore (as $\epsilon \to 0$)

$$\frac{1}{x \pm i\epsilon} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

There is another interesting fact we can derive. If a sequence f_n of functions tends to a generalized function f then we say that the sequence f'_n of derivatives of the functions defines the **derivative** f' of the generalized function. You can check in this way that the derivative of the step function is the Dirac delta function :

$$\theta'(x) = \delta(x)$$

§ 9

Indirect definition

The method of indirect definition of a generalized function is somewhat like the way police obtains information on hard criminals through its informers who are themselves better behaved but happen to be in the company of the those wanted men.

In this method the effect of the generalized function is seen when "it is smeared with a test function". This means we integrate the generalized function to be defined with a known well behaved "test" function ϕ and give the value of the integral

$$(f, \phi) \equiv \int_{-\infty}^{+\infty} f(x)\phi(x)dx$$

The class $\mathcal{D}$ of test functions should be sufficiently large so that a knowledge of (f, ϕ) for all $\phi \in \mathcal{D}$ is enough to extract all knowledge of the generalized function f.

The class of functions $\mathcal{D}$ is taken to be the set of all functions which are infinitely differentiable and which vanish outside a finite interval.

[Example and remark about the difference between analytic real, and analytic complex functions.]

It is hoped that for functions with a singularity, the process of integrating with a very well behaved and smooth function ϕ will give meaningful result (f, ϕ), even though it may not be possible to define the function at all points by the usual definition.

For example, for our step function, the definition as a generalized function is

$$(\theta, \phi) = \int_0^{+\infty} \phi(x) dx$$

which is obvious in this simple case.

What is not obvious is that this definition can be used to define a derivative of the step function which is another generalized function.

If there was a normal function $f(x)$ we would have written for its derivative $f' = df/dx$

$$(f', \phi) = \int_{-\infty}^{+\infty} f'(x)\phi(x) dx = [f\phi]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(x)\phi'(x) dx = -(f, \phi')$$

the first term being zero because every ϕ vanishes at $\pm\infty$.

We use the above relation to define the derivative f' of a generalised function f,

$$(f', \phi) = -(f, \phi')$$

As an example we can define the derivative of the step function θ as

$$(\theta', \phi) = -(\theta, \phi') = \int_0^{+\infty} \phi'(x) dx = -[\phi(x)]_0^{\infty} = \phi(0)$$

Therefore θ' is a new generalized function, traditionally denoted by δ and called **Dirac delta function**, (which Dirac had defined in 1930 to replace the Kronecker delta δ_{ij} in quantum mechanics for the continuous case $\delta(x-y)$).

$$(\delta, \phi) = \phi(0)$$

If we try to find the values of $\theta'(x) = \delta(x)$, it is zero for every $x < 0$ or $x > 0$. This is obvious because in these places the function θ is a constant. The new thing about the definition of generalized function is the derivative (a generalized derivative). We expect the derivative to go to infinity, because of the finite jump in the step function at $x = 0$. The usual way to define delta function is to say that it is a function which is zero everywhere except at $x = 0$ where it is infinity in such a way that the integral of the function ("area under the curve") is unity

$$\int \delta(x)dx = 1$$

What is more δ, being a generalized function itself, has its own derivative defined

$$(\delta', \phi) = -(\delta, \phi') = -\phi'(0)$$

Thus we see that generalized functions have derivatives of all orders defined – which is good progress considering that it was not possible to differentiate them even once by by the usual definition.

§ 10

Fourier Transform of δ

We shall prove a very important formula, which can be written as

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx}\, dk$$

As things stand in this formula the integral on the right hand side is not well defined. This formula is very useful but symbolic. One way is to define it as the limit of convergent integrals

$$\begin{aligned} \delta(x) &= \lim_{\epsilon \to 0} \delta_\epsilon(x) \\ &\equiv \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx - \epsilon k^2}\, dk \end{aligned}$$

The integral can be done explicitly by "completing the square"

$$\begin{aligned} -\epsilon k^2 + ikx = -\epsilon \left(k^2 - \frac{ikx}{\epsilon} \right) &= -\epsilon \left[\left(k - \frac{ix}{2\epsilon} \right)^2 - \left(\frac{ix}{2\epsilon} \right)^2 \right] \\ &= -\epsilon K^2 - \frac{x^2}{4\epsilon} \end{aligned}$$

where $K = k - ikx/(2\epsilon)$. The integration variable can be changed from k to K and the integral evaluated

$$\begin{aligned} \delta_\epsilon(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx - \epsilon k^2}\, dk \\ &= \frac{e^{-x^2/4\epsilon}}{2\pi} \int_{-\infty}^{+\infty} e^{-\epsilon K^2}\, dK \\ &= \frac{e^{-x^2/4\epsilon}}{2\sqrt{\pi\epsilon}} \end{aligned}$$

These functions give us another insight into the generalized functions like the delta function. For each finite value of the positive number ϵ the function is a gaussian curve with a width proportional to $\sqrt{\epsilon}$ and a height proportional to $1/\sqrt{\epsilon}$. The "area under the curve", that is $\int \delta(x)dx$ is always unity, for any ϵ. As $\epsilon \to 0$, the functions δ_ϵ become more and more narrow and the peak higher and higher at $x = 0$. The Dirac delta function is the "singular limit" of such a sequence of functions.

§ 11

Differential equations in generalized functions

Consider the simple differential equation

$$\left(\frac{d}{dt} + a\right) F = \delta(t)$$

We already know that $\theta'(t) = \delta(t)$. Therefore we try a solution of the type $F(t) = \theta(t)f(t)$ where f is an normal unknown function. Substituting we get

$$\begin{aligned} \delta(t)f(t) + \theta(t)f(t) + a\theta(t)f(t) &= \delta(t)f(0) + \theta(t)f'(t) + a\theta(t)f(t) \\ &= \delta(t) \end{aligned}$$

This implies that $f(0) = 1$ and $f'(t) = -af(t)$ for $t > 0$. Therefore the solution is

$$F(t) = \theta(t)e^{-at}$$

we will need this formula for calculating the Green's function for the heat equation.

Lecture 3.4
Green's Functions

Whenever there is an inhomogeneous PDE, the Dirac delta function can be used to model a "point source". The solution for a point source is called "fundamental solution" or a **Green's function**. The solution for a general source can be then constructed using the Green's function.

§ 12

An important formula

First let us introduce the three dimensional Dirac delta function $\delta^3(\mathbf{r})$,

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}.\mathbf{r}}\, d^3\mathbf{k}$$

The function has value zero at all points except the origin $\mathbf{r} = 0$, and is such that for any smooth function ϕ of $\mathbf{r}$, the integral

$$\int d^3\mathbf{r}\delta(\mathbf{r})\phi(\mathbf{r}) = \phi(0)$$

In particular we can write

$$\int_V d^3\mathbf{r}\delta(\mathbf{r}) = 1$$

where V is a region including the origin. The integral above is zero when the region V excludes the origin $\mathbf{r} = 0$. This shows, for example, that $q\delta(\mathbf{r})$ represents a charge density of a point particle with charge q.

The Poisson equation of electrostatics

$$\nabla^2 \Phi = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$

gives the electrostatic potential $\Phi(\mathbf{r})$ for a charge distribution $\rho(\mathbf{r})$.

The method of Green's function involves solving the equation

$$\nabla^2 G(\mathbf{r} - \mathbf{r}') = \delta^3(\mathbf{r} - \mathbf{r}')$$

Suppose we are able to solve this equation, then the solution to our original can be written as

$$\Phi(\mathbf{r}) = -\frac{1}{\epsilon_0} \int d^3\mathbf{r}' G(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}')$$

because

$$\begin{aligned} \nabla^2 \int d^3\mathbf{r}' G(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') &= \int d^3\mathbf{r}' \nabla^2 G(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') \\ &= \int d^3\mathbf{r}' \delta^3(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') \\ &= \rho(\mathbf{r}) \end{aligned}$$

Now for the construction of the Green's function. We verify that

$$\nabla^2 \left(-\frac{1}{4\pi r} \right) = \delta^3(\mathbf{r})$$

This is a very important formula.

We see immediately that (by expressing ∇^2 in polar coordinates $\nabla^2(1/r) = 0$ for any $r \neq 0$. What we nust verify therefore to prove that the right hand side is indeed a delta function,

$$\int_V d^3\mathbf{r}\, \nabla^2 \left(-\frac{1}{4\pi r}\right) = 1$$

if the volume V includes the origin. Let us take the region V to be a sphere of radius a centered at the origin. Then using the Gauss theorem

$$\int_V d^3\mathbf{r}\, \nabla . \left[\nabla \left(-\frac{1}{4\pi r}\right)\right] = -\frac{1}{4\pi} \int_S \nabla \left(\frac{1}{r}\right) .\mathbf{n}\, dS$$

where $\mathbf{n}$ is the unit radial vector on spherical surface S. Using the fact that at the boundary S of V where $r \neq 0$,

$$\nabla \left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3} = -\frac{\mathbf{n}}{r^2}$$

we get

$$\frac{1}{4\pi} \int_S \frac{\mathbf{n}.\mathbf{n}}{r^2} dS = 1$$

which proves the result.

§ 13

Fourier Transform Method

We now introduce the method to determine the Green's function directly by the powerful Fourier transform method. Let,

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\, \tilde{G}(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}$$

Then, applying ∇^2

$$\nabla^2 G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\, (-\mathbf{k}^2) \tilde{G}(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}$$

and equating it to the Fourier expansion of the delta function

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k}\, e^{i\mathbf{k}.\mathbf{r}}$$

we get

$$\tilde{G}(\mathbf{k}) = -\frac{1}{\mathbf{k}^2}$$

So the Greens function is

$$G(\mathbf{r}) = \frac{-1}{(2\pi)^3} \int d^3\mathbf{k}\, \frac{1}{\mathbf{k}^2}\, e^{i\mathbf{k}.\mathbf{r}}$$

Provided we can integrate the right hand side. The right hand side is meaningless as it stands because it is not defined at $\mathbf{k} = 0$. We introduce a small positive number μ and interpret the integral as the limit $\mu \to 0$ of

$$G_\mu(\mathbf{r}) = \frac{-1}{(2\pi)^3} \int d^3\mathbf{k}\, \frac{1}{\mathbf{k}^2 + \mu^2}\, e^{i\mathbf{k}.\mathbf{r}}$$

The integral on the right hand side in independent of what directions are chosen for the axes in the k-space. For the fixed value of $\mathbf{r}$ we chose the direction of k_z axis along $\mathbf{r}$. Then calling the polar coordinates in the k-space by k, θ, ϕ we get $\mathbf{k}.\mathbf{r} = kr\cos\theta$, and integrating the ϕ and calling $\cos\theta = u$

$$\begin{aligned} G_\mu(\mathbf{r}) &= \frac{-1}{(2\pi)^2} \int_0^\infty k^2\, dk \int_{-1}^1 du\, \frac{e^{ikru}}{k^2+\mu^2} \\ &= \frac{-1}{ir(2\pi)^2} \int_0^\infty k\, dk\, \frac{e^{ikr} - e^{-ikr}}{k^2+\mu^2} \\ &= \frac{-1}{ir(2\pi)^2} \int_{-\infty}^\infty k\, dk\, \frac{e^{ikr}}{k^2+\mu^2} \end{aligned}$$

The integral over k can be interpreted as a complex k-integration along a closed contour running along the real axis from $-\infty$ to ∞ and along an infinite semicircle in the upper half plane because on this part of the contour the integrand is zero ($\exp(i(k + i\eta)r) = \exp(-\eta r + ikr) \to 0$ for large η. There is one pole in the upper half plane, at $k = i\mu$ (because $\mu > 0$ by definition). We write

$$k\frac{e^{ikr}}{k^2 + \mu^2} = \frac{e^{ikr}}{2}\left[\frac{1}{k + i\mu} + \frac{1}{k - i\mu}\right]$$

and the residue theorem gives

$$\begin{aligned} G_\mu(\mathbf{r}) &= \frac{-1}{ir(2\pi)^2}\int_{-\infty}^{\infty} k\,dk\,\frac{e^{ikr}}{k^2 + \mu^2} \\ &= \frac{-1}{ir(2\pi)^2}(2\pi i)\frac{e^{-\mu r}}{2} \\ &= -\frac{e^{-\mu r}}{4\pi r}. \end{aligned}$$

This result is itself of importance for finite μ and it is called the Yukawa potential. The Yukawa potential is the Green's function for the inhomogeneous equation

$$(\nabla^2 - \mu^2)G(\mathbf{r}) = \delta(\mathbf{r}), \qquad \mu > 0.$$

For $\mu \to 0$ we recover the result for the Poisson equation:

$$G(\mathbf{r}) = \lim_{\mu \to 0} -\frac{e^{-\mu r}}{4\pi r} = -\frac{1}{4\pi r}$$

§ 14

Green's function for the heat equation

In this case we have a function on both $\mathbf{r}$ and t.

The equation to be solved is

$$\frac{\partial G(\mathbf{r}, t)}{\partial t} = D\nabla^2\, G(\mathbf{r}, t) + \delta^3(\mathbf{r})\delta(t)$$

We introduce the fourier transform of $G(\mathbf{r},t)$ with respect to $\mathbf{r}$:

$$G(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,\tilde{G}(\mathbf{k},t)\,e^{i\mathbf{k}.\mathbf{r}}$$

and write the delta function $\delta^3(\mathbf{r})$ by its usual formula

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,e^{i\mathbf{k}.\mathbf{r}}$$

we find after substituting in the heat equation and comparing both sides

$$\frac{\partial\tilde{G}(\mathbf{k},t)}{\partial t} = -D\mathbf{k}^2\tilde{G}(\mathbf{k},t) + \delta(t)$$

which can be solved using the example of section §11:

$$\tilde{G}(\mathbf{k},t) = \theta(t)\exp[-D\mathbf{k}^2 t]$$

therefore

$$G(\mathbf{r},t) = \frac{\theta(t)}{(2\pi)^3}\int d^3\mathbf{k}\,\exp[-D\mathbf{k}^2 t]\,e^{i\mathbf{k}.\mathbf{r}}$$

We can perform the three dimensional integral on $\mathbf{k}$ by noting that there are actually three independent gaussian integrals

$$\begin{aligned}\int d^3\mathbf{k}\,\exp[-D\mathbf{k}^2 t]\,e^{i\mathbf{k}.\mathbf{r}} &= \int dk_x\,\exp[-Dk_x^2 t + ik_x x]\int dk_y\,\exp[-Dk_y^2 t + ik_y y] \\ &\times \int dk_z\,\exp[-Dk_z^2 t + ik_z z]\end{aligned}$$

The first integral gives for example

$$\int dk_x\,\exp[-Dk_x^2 t + ik_x x] = \sqrt{\frac{\pi}{Dt}}\exp[-x^2/(4Dt)]$$

and similarly for the other two factors.

The result is

$$G(\mathbf{r},t) = \frac{\theta(t)}{(2\sqrt{\pi Dt})^3}\,\exp[-\mathbf{r}^2/(4Dt)]$$

Lecture 3.5
Green's Function for the Helmholtz and the Wave Equation

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§ 15

Green's function for the Helmholtz equation

The equation to be solved is

$$(\nabla^2 + \kappa^2)G(\mathbf{r}) = \delta(\mathbf{r}),$$

where κ is a *positive* real number. As in the case of Poisson equation we assume

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,\tilde{G}(\mathbf{k})\,e^{i\mathbf{k}.\mathbf{r}}$$

and obtain

$$\tilde{G}(\mathbf{k}) = -\frac{1}{k^2 - \kappa^2}, \qquad k = |\mathbf{k}|.$$

In the Poisson case, $\tilde{G}(\mathbf{k}) = -1/k^2$ gave a problem at $k^2 = 0$ and we redefined the integral for $G(\mathbf{r})$ by changing k^2 to $k^2 + \mu^2$ with $\mu \to 0$. Here, that trick will not do because the singularity cannot be removed that way. (Figure out why?)

In the present case we avoid the singularity at $k = \pm\kappa$ by making κ complex by adding an infinitesimal imaginary part. We can do this in two ways, adding a positive or a negative imaginary part. Both ways are equally valid and give two independent Green's functions.

$$\tilde{G}_{\pm}(\mathbf{k}) = \lim_{\epsilon \to 0} -\frac{1}{k^2 - \kappa_\epsilon^2}, \qquad \kappa_\epsilon = \kappa \pm i\epsilon.$$

The remaining calculation is exactly as that for the Poisson case, (do that), thus

$$G_\pm(\mathbf{r}) = -\frac{e^{\pm i\kappa r}}{4\pi r}.$$

§ 16

Green's function for the Wave equation

The fundamental equation for the wave equation is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t) = \delta(t)\delta(\mathbf{r}).$$

Traditionally we do it in two steps. First, we define the time-Fourier transform of G

$$G(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{G}(\mathbf{r}, \omega) e^{i\omega t}$$

which shows that the intermediate transform $\hat{G}$ satisfies the Helmholtz equation

$$(\nabla^2 + \kappa^2)\hat{G}(\mathbf{r}, \omega) = \delta(\mathbf{r}), \qquad \kappa = \omega/c$$

because on the right hand side $\delta(t)$ can be written as Fourier transform

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t},$$

and comparing. So, using the solution for the Helmholtz equation, we write

$$\hat{G}_\pm(\mathbf{r}, \omega) = -\frac{e^{\pm i\omega r/c}}{4\pi r}.$$

Next, this value of $\hat{G}$ can be substituted to calculate the Green's function

$$\begin{aligned} G_{\pm}(\mathbf{r}, t) &= -\frac{1}{4\pi r}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t \pm r/c)} \\ &= -\frac{1}{4\pi r}\delta\left(t \pm \frac{r}{c}\right). \end{aligned}$$

The two solutions; one with $\delta(t - r/c)$ is called the *retarded* and the one with $\delta(t + r/c)$ is called the *advanced* solution. The reason for these names will appear in the next sections.

Lecture 3.6
Fields for Arbitrary Charge and Current Densities

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§ 17

The wave equations

The Maxwell equations imply wave equations for electric and magnetic fields. Taking 'curl' $\nabla \times (\nabla \times \mathbf{E})$ of the Faraday's law of induction gives

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

where we have interchanges the time and space derivative on the right hand side. Substituting for $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{B}$ from the other Maxwell equations we get

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \frac{1}{\epsilon_0}\nabla\rho + \frac{1}{\epsilon_0 c^2}\frac{\partial \mathbf{j}}{\partial t} \tag{36}$$

Similarly taking 'curl' of $\nabla \times \mathbf{B}$ we can get

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = -\frac{1}{\epsilon_0 c^2}\nabla \times \mathbf{j} \tag{37}$$

These are inhomogeneous wave equations whose particular solution are known to us through the Green's function. The solution of

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{r}, t) = \psi(\mathbf{r}, t) \tag{38}$$

is

$$\begin{aligned}\phi(\mathbf{r},t) &= \int d^3\mathbf{r}' \int dt' G(\mathbf{r}-\mathbf{r}', t-t')\psi(\mathbf{r}',t') \qquad (39)\\ &= -\int d^3\mathbf{r}' \int dt' \frac{1}{4\pi R}\delta(t-t'-R/c)\psi(\mathbf{r}',t')\\ &= -\int d^3\mathbf{r}' \frac{1}{4\pi R}\psi(\mathbf{r}', t-R/c)\end{aligned} \qquad (40)$$

where we have substituted the Green's function for the wave equation given by the expression

$$G(\mathbf{r}-\mathbf{r}', t-t') = -\frac{1}{4\pi R}\delta(t-t'-R/c), \qquad R = |\mathbf{r}-\mathbf{r}'|. \qquad (41)$$

as calculated in lecture 3.4.

Our equations (1) and (2) are of the same from as (3) but they also involve *derivatives*. And handling derivatives requires a little care for the following reason :

The right hand side is an integral over whatever the function ψ is evaluated with t replaced by $t - |\mathbf{r}-\mathbf{r}'|/c$. The function $\psi(\mathbf{r}', t')$ which was a function of one vector variable $\mathbf{r}'$ and one scalar t' respectively has become a function of $\mathbf{r}', \mathbf{r}$ and t. If we had $\partial\psi(\mathbf{r}',t')/\partial x'$, for example, on the right hand side in place of ψ then

$$\left.\frac{\partial\psi(\mathbf{r}',t')}{\partial x'}\right|_{t'=t-R/c} \qquad (42)$$

would be the quantity inside the integral sign. Which is, bye the way, not the same thing as

$$\frac{\partial\psi(\mathbf{r}', t-R/c)}{\partial x'} \qquad (43)$$

because of the additional dependence on $\mathbf{r}'$ through R. Another way to say this is that while $\partial/\partial x'$ in (7) means differentiating with respect to x' keeping t' constant, in (8) it means differentiating with respect to x' keeping t and $\mathbf{r}$ constant.

You can ask why we should be bothered to use (8) when (7) is already the correct expression? The answer is that we would like to use one quantity $\psi(\mathbf{r}', t-R/c)$ and its derivatives rather than two quantities $\psi(\mathbf{r}', t-R/c)$ and $[\nabla'\psi(\mathbf{r}', t')]_{t'=t-R/c}$.

The two are related of course.

$$\frac{\partial\psi(\mathbf{r}', t-R/c)}{\partial x'} = \left.\frac{\partial\psi(\mathbf{r}', t')}{\partial x'}\right|_{t'=t-R/c} + \left.\frac{\partial\psi(\mathbf{r}', t')}{\partial t'}\right|_{t'=t-R/c} \frac{\partial}{\partial x'}\left(\frac{-R}{c}\right)$$

Similarly, if we had a time derivative $\partial\psi(\mathbf{r}', t')/\partial t'$, then

$$\left.\frac{\partial\psi(\mathbf{r}', t')}{\partial t'}\right|_{t'=t-R/c}$$

which occurs inside the integral over $d^3\mathbf{r}'$ is related to the derivative w.r.t. t more simply by

$$\frac{\partial\psi(\mathbf{r}', t-R/c)}{\partial t} = \left.\frac{\partial\psi(\mathbf{r}', t')}{\partial t'}\right|_{t'=t-R/c}$$

After this explanation we can proceed to write our solutions.

§ 18

Jefimenko's expressions for E and B

For the electric field the right hand side involves $\nabla'\rho(\mathbf{r}', t')$ and $\partial\mathbf{j}(\mathbf{r}', \mathbf{t}')/\partial t'$ both evaluated at $t' = t - R/c$. Using

$$\nabla'\rho(\mathbf{r}', t-R/c) = \nabla'\rho(\mathbf{r}', t')|_{t'=t-R/c} + \left.\frac{\partial\rho(\mathbf{r}', t')}{\partial t'}\right|_{t'=t-R/c} \nabla'\left(\frac{-R}{c}\right)$$

and

$$\nabla' R = \nabla'|\mathbf{r} - \mathbf{r}'| = -\frac{\mathbf{R}}{R} \equiv -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

and an integration by part

$$\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx' \frac{\nabla' f(\mathbf{r}')}{R} = -\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \, \nabla' \left(\frac{1}{R}\right) f(\mathbf{r}')$$

we get (do the algebra!)

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{r}' \Big[\frac{\mathbf{R}}{R^3} \rho(\mathbf{r}', t - R/c) + \frac{\mathbf{R}}{cR^2} \frac{\partial \rho(\mathbf{r}', t - R/c)}{\partial t} \\ &\quad - \frac{1}{c^2 R} \frac{\partial \mathbf{j}(\mathbf{r}', t - R/c)}{\partial t} \Big] \end{aligned} \tag{44}$$

Similarly, the magnetic induction field is obtained as

$$\mathbf{B}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0 c^2} \int d^3\mathbf{r}' \Big[\frac{\mathbf{R}}{R^3} \times \mathbf{j}(\mathbf{r}', t - R/c) + \frac{\mathbf{R}}{cR^2} \times \frac{\partial \mathbf{j}(\mathbf{r}', t - R/c)}{\partial t} \Big] \tag{45}$$

These expressions for **E** and **B** are called Jefimenko's equations.

Lecture 3.7
Fields from a Point Charge

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§ 19

Lienard-Wiechert Potentials

The potentials due to a **point charge** were calculated by Lienard (1898) and Wiechert (1900).

Recall (from lecture 3.2) that the potentials in the Lorentz gauge ($\nabla \cdot \mathbf{A} + \partial\phi/(c^2\partial t)$) satisfy the wave equations,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{r},t) = -\frac{\rho(\mathbf{r},t)}{\epsilon_0} \tag{46}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) = -\frac{\mathbf{j}(\mathbf{r},t)}{\epsilon_0 c^2} \tag{47}$$

The solutions of these using the Green's function for the wave equation are

$$\begin{aligned}
\phi(\mathbf{r},t) &= -\frac{1}{\epsilon_0}\int d^3\mathbf{r}'\int dt' G(\mathbf{r}-\mathbf{r}', t-t')\rho(\mathbf{r}',t') \qquad (48)\\
&= \frac{1}{4\pi\epsilon_0}\int d^3\mathbf{r}'\int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)\rho(\mathbf{r}',t')\\
&= \frac{1}{4\pi\epsilon_0}\int d^3\mathbf{r}' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/c)
\end{aligned}$$

If the point particle with charge q moves along a trajectory so that at time t the position of the point particle is $\mathbf{s}(t)$ then the charge density is

$$\rho(\mathbf{r},t) = q\delta^3(\mathbf{r}-\mathbf{s}(t)). \tag{49}$$

The potential due to this charge is

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int d^3\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta^3(\mathbf{r}' - \mathbf{s}(t - |\mathbf{r} - \mathbf{r}'|/c))$$

Eveluation of this integral requires some care because the argument of the Dirac delta functions (there are three of them) are not in the variables $\mathbf{r}' = (x^{1'}, x^{2'}, x^{3'})$ but more complicated functions :

$$\delta^3(\mathbf{r}' - \mathbf{s}(t - |\mathbf{r} - \mathbf{r}'|/c)) \equiv \delta^3(\mathbf{f})$$

where

$$\mathbf{f}(\mathbf{r}, \mathbf{r}', t) = \mathbf{r}' - \mathbf{s}(t - |\mathbf{r} - \mathbf{r}'|/c)$$

Actually $\mathbf{r}$ and t are 'sitting' variables doing nothing as far as the integration is concerned. The real functional relationship is between $\mathbf{r}'$ and $\mathbf{f}$. The integration can be changed to variables $\mathbf{f}$ in place of $\mathbf{r}'$

$$\begin{aligned} d^3\mathbf{r}' &= d^3\mathbf{f} \times \frac{1}{J} \\ J &= \det \left\| \frac{\partial f^i}{\partial x'^j} \right\| \end{aligned}$$

The Jacobian matrix can be easily calculated : for example

$$\begin{aligned} \frac{\partial f_i}{\partial x'^j} &= \delta^i_j - \frac{ds^i}{dt} \frac{\partial}{\partial x'^j} \left(\frac{-|\mathbf{r} - \mathbf{r}'|}{c} \right) \\ &= \delta^i_j - \frac{ds^i}{dt} \frac{x^j - x'^j}{c|\mathbf{r} - \mathbf{r}'|} \\ &= \delta^i_j - \frac{v^i n^j}{c} \end{aligned}$$

where $\mathbf{v} = d\mathbf{s}/dt$ is the velocity of the particle and $\mathbf{n} = (\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ is the unit vector pointing from $\mathbf{r}'$ towards $\mathbf{r}$. You can check that the determinant of this matrix is

$$J = 1 - \mathbf{v} \cdot \mathbf{n}/c$$

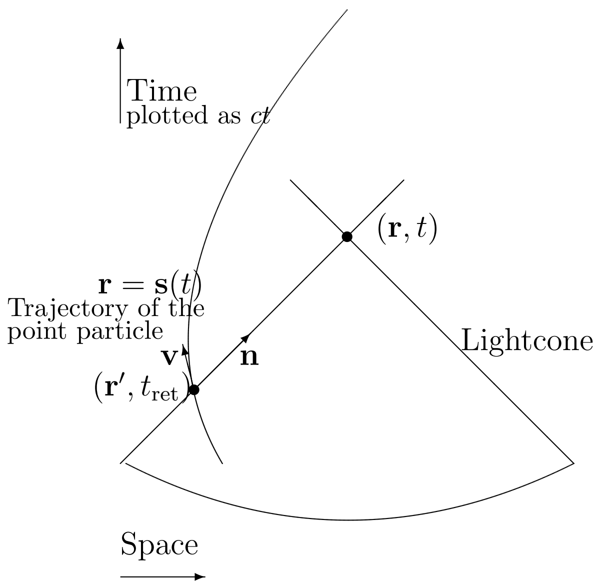
Therefore,

$$
\begin{aligned}
\phi(\mathbf{r},t) &= \frac{q}{4\pi\epsilon_0}\int d^3\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta^3(\mathbf{r}'-\mathbf{s}(t-|\mathbf{r}-\mathbf{r}'|/c)) \\
&= \frac{q}{4\pi\epsilon_0}\int d^3\mathbf{f}\,\delta^3(\mathbf{f})\frac{1}{J}\frac{1}{|\mathbf{r}-\mathbf{r}'(\mathbf{f})|} \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{J}\frac{1}{|\mathbf{r}-\mathbf{r}'(\mathbf{f})|}\right]_{\mathbf{f}=\mathbf{0}} \\
&= \frac{q}{4\pi\epsilon_0}\times\frac{1}{1-\mathbf{v}(t_{\rm ret})\cdot\mathbf{n}/c}\times\frac{1}{R}, \qquad R=|\mathbf{r}-\mathbf{s}(t_{\rm ret})|
\end{aligned}
$$

Here the solution of

$$\mathbf{f}=\mathbf{r}'-\mathbf{s}(t-|\mathbf{r}-\mathbf{r}'|/c)=0$$

for $\mathbf{r}'$ for a fixed value of $\mathbf{r}$ and t is obtained as follows : Start from the space-time point $\mathbf{r},t$. Draw the past light cone from this point. Find the space-time point at which the trajectory of the particle $\mathbf{s}(t)$ intersects the past light cone. This point is precisely $\mathbf{r}',t_{\rm ret}$. (See the diagram.) The value of $\mathbf{r}'$ is the position of the charged point particle at $t_{\rm ret}$. So $\mathbf{r}-\mathbf{r}'$ should be replaced by $\mathbf{r}-\mathbf{s}(t_{\rm ret})$ when integration over $\delta^3(\mathbf{f})$ is performed.



You should appreciate the elegance and convenience of the Dirac delta function in getting the Lienard-Wiechert potential by comparing to the original proof which was given in 1900 and is summarized in the book *History of theories of Aether and Electricity* by E. T. Whittaker.

Lecture 3.8
Radiation from an accelerated charge

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§ 20

Electromagnetic potentials

We have seen (Lienard-Wiechert potentials) that the potentials ϕ and $\mathbf{A}$ at a point $\mathbf{r}$ and time t due to a charge q located at $\mathbf{r}'$ very close to the origin (at the retarded time) are

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(\frac{1}{1 - [\mathbf{v}].\mathbf{n}/c}\right) \tag{50}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0 c^2}\frac{1}{r}\left(\frac{[\mathbf{v}]}{1 - [\mathbf{v}].\mathbf{n}/c}\right) \tag{51}$$

where $r = |\mathbf{r} - \mathbf{r}'| \approx |\mathbf{r}|$,
$\mathbf{n}$ is the unit vector in the direction of $\mathbf{r}$, and
$[\mathbf{v}]$ is the velocity $\mathbf{v} = d\mathbf{r}'/dt'$ of the particle at the retarded time $t' = t - r/c$.

When the charged particle is moving with a velocity small compared to velocity of light, the term $[\mathbf{v}].\mathbf{n}/c$ can treated as a small quantity (compared to 1), and we can keep it only upto its first order in equations,

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 + \frac{[\mathbf{v}].\mathbf{n}}{c}\right) \tag{52}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0 c^2}\frac{[\mathbf{v}]}{r} \tag{53}$$

In these expressions the dependence on time t comes indirectly through

$$[\mathbf{v}] = \mathbf{v}(t - r/c).$$

Also note that any function of the form

$$F(r,t) \equiv \frac{f(t-r/c)}{r}$$

automatically satisfies the wave equation,

$$\begin{aligned} \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)F(r,t) &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)F(r,t) \\ &= 0. \end{aligned}$$

Therefore both ϕ and $\mathbf{A}$ in (52) and (53) satisfy the wave equation far away from the origin $r = 0$.

§ 21

Electromagnetic fields

We now calculate electromagnetic fields given by the expressions

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{54}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{55}$$

In order to get simpler formulas, we take the case of a charged particle moving only along the z-axis near the origin. Then $\mathbf{v} = (0,0,v)$ and $\mathbf{v}.\mathbf{n} = vz/r$.

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} + \frac{q}{4\pi\varepsilon_0}\frac{vz}{r^2c} \tag{56}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{rc^2}(0,0,v), \tag{57}$$

where it is understood that v is a function of $t - r/c$.

Furthermore, as $1/r$ is small (we call the region where r is large as 'radiation zone') we drop all field terms which are of order $O(1/r^2)$. The leading $O(1/r)$ terms come only from differentiating the velocity $v = v(t - r/c)$. We also omit the common factor $q/4\pi\varepsilon_0$ and only include it in the final expressions. The electric field has two terms

$$
\begin{aligned}
-\nabla\phi &= O(1/r^2) + \frac{az}{r^3c^2}\mathbf{r} \\
-\frac{\partial \mathbf{A}}{\partial t} &= \left(0, 0, -\frac{a}{rc^2}\right),
\end{aligned}
$$

where $a = dv/dt'$ is the acceleration of the charge at the retarded time. Similarly, we calculate the magnetic field $B = \nabla\times\mathbf{A}$. Thus the fields in the radiation zone (omitting $O(1/r^2)$ terms) are

$$
\begin{aligned}
\mathbf{E}_{\text{rad}} &= \frac{q}{4\pi\varepsilon_0}\frac{a}{rc^2}\left(\frac{xz}{r^2}, \frac{yz}{r^2}, \frac{z^2}{r^2} - 1\right) &(58)\\
\mathbf{B}_{\text{rad}} &= \frac{q}{4\pi\varepsilon_0}\frac{a}{rc^3}\left(-\frac{y}{r}, \frac{x}{r}, 0\right). &(59)
\end{aligned}
$$

We notice that $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$ are orthogonal, and perpendicular to the radial vector $\mathbf{r}$. In fact if we use the polar coordinates (r, θ, ϕ) then the $\mathbf{B}_{\text{rad}}$ lines of force are circles of constant θ and r with increasing angle ϕ , whereas $\mathbf{E}_{\text{rad}}$ lines are circles of constant ϕ and r but increasing θ.

The energy radiated in the radial direction $\mathbf{n} = \mathbf{r}/r$ is given by the Poynting vector

$$
\mathbf{S}_{\text{rad}} = \varepsilon_0 c^2 \mathbf{E}_{\text{rad}} \times \mathbf{B}_{\text{rad}} = \left(\frac{q^2}{16\pi^2\varepsilon_0}\right)\frac{a^2\sin^2\theta}{r^2c^3}\mathbf{n}. \qquad (60)
$$

The total energy radiated per unit time is obtained by integrating this expression over the surface of a sphere. That integral is called J. J. Larmor's formula

$$
S_{\text{tot}} = \frac{q^2a^2}{6\pi\varepsilon_0 c^3} = \frac{q^2}{4\pi\varepsilon_0}\frac{2}{3}\frac{a^2}{c^3}. \qquad (61)
$$