QFT Solved Problem
Dyson Expansion
Second Order Term

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Abstract

The n^{th} order term, in the perturbation series for time evolution operator in the interaction picture, can be written in a symmetric fashion as $1/n!\times$ a time ordered product. This comes by suitable by manipulating the multiple integral as demonstrated for the second order term.

Problem 1: Prove that

$$\int_{t_0}^{T} dt_1 \int_{t_0}^{t_1} dt_2\, H_I'(t_1)H_I'(t_2) = \frac{1}{2}\int_{t_0}^{T} dt_1 \int_{t_0}^{T} dt_2 T\Big(H_I'(t_1)H_I'(t_2)\Big) \tag{1}$$

Solution Consider the right and side

$$\int_0^T dt_1 \int_0^T dt_2 T\Big(H(t_1)H(t_2)\Big) \tag{2}$$

$$\text{We split } \int_0^T dt_2 \longrightarrow \int_0^{t_1} dt_2 + \int_{t_1}^T dt_2 \tag{3}$$

$$= \int_0^T dt_1 \int_0^{t_1} dt_2 T\Big(H(t_1)H(t_2)\Big) + \int_0^T dt_1 \int_{t_1}^T dt_2 T\Big(H(t_1)H(t_2)\Big) \tag{4}$$

$$= \int_0^T dt_1 \int_0^{t_1} dt_2 \Big(H(t_1)H(t_2)\Big) + \int_0^t dt_1 \int_{t_1}^T dt_2 \Big(H(t_2)H(t_1)\Big). \tag{5}$$

We have removed the time ordering operator and explicitly written the product with correct ordering. In the second term we reverse the order of integration from $\iint dt_1 dt_2 \to \iint dt_2 dt_1$. For this purpose it should be noted that the second integral is over the half of the square OAB in the (t_1, t_2) plane as shown in Fig. 1(a) below.

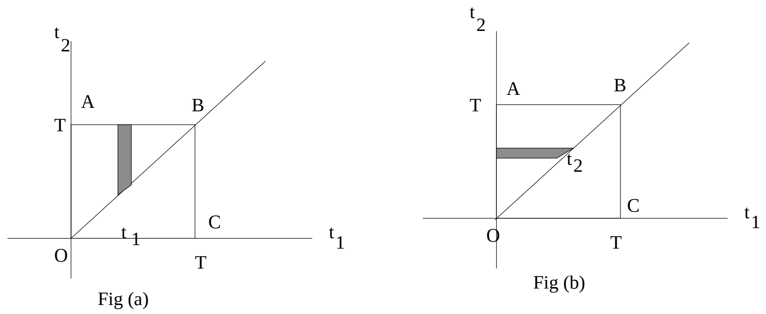

Fig. **1**

In Fig 1(a) the upper rectangle OAB is covered by fixing t_2 in the range taking $0 < t_2 < T$ and varying t_1 in the range $t_1 < t_1 < T$. In Fig 1(b) the same area OAB is covered by interchanging the order. Thus we first choose t_2 in the range $0 < t_2 < T$ and vary t_1 in the range $0 < t_1 < t_2$. Thus we have the following rule for change of order of integration.

$$\int_0^T dt_1 \int_{t_1}^T dt_2 \to \int_0^T dt_2 \int_0^{t_2} dt_1 \tag{6}$$

Using this rule to interchange the order of integration in (8), we get

$$\int_0^T dt_1 \int_0^t dt_2 T\Big(H(t_1)H(t_2)\Big) \tag{7}$$

$$= \int_0^T dt_1 \int_0^{t_1} dt_2 \Big(H(t_1)H(t_2)\Big) + \int_0^T dt_1 \int_0^T dt_2 \Big(H(t_2)H(t_1)\Big). \tag{8}$$

$$= \int_0^T dt_1 \int_0^{t_1} dt_2 \Big(H(t_1)H(t_2)\Big) + \int_0^T dt_2 \int_0^{t_2} dt_1 \Big(H(t_2)H(t_1)\Big). \tag{9}$$

We now exchange the labels of integration variables $(t_1 \to t_2)$ in the last term to get

$$\int_0^T dt_1 \int_0^t dt_2 T\Big(H(t_1)H(t_2)\Big) = 2\int_0^T dt_1 \int_0^{t_1} dt_2 \Big(H(t_1)H(t_2)\Big) \tag{10}$$

which proves the desired result.