Quantum Field Theory 2016- Final Examination

Q1(b) Compute the matrix element

$$\langle s, \vec{q}|J_\mu(x)|r, \vec{p}\rangle$$

where $J_\mu(x)$ is the current for a Dirac particle and $|s, \vec{q}\rangle$ and $|r, \vec{p}\rangle$ denote one particle states of spin half particle with spin and momenta as specified. [10+10]

Solution: Using notation of Gasiorowic

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int \sum_r \frac{d^3k}{(E_k/M)} \Big\{ e^{-ikx} a^{(r)}(k) u^{(r)}(k) + b^{(r)+}(k) v^{(r)}(k) e^{ikx} \Big\}$$

and

$$\bar{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \sum_s \int \frac{d^3p}{(E_k/M)} \Big\{ e^{ikx} a^{(r)+}(k) \bar{u}^{(r)}(k) + b^{(r)}(k) \bar{v}^{(r)}(k) e^{-ikx} \Big\}$$

$|r\vec{p}\rangle = a^{+(r)}(p)|0\rangle$ and $|s\vec{q}\rangle = \overset{(s)+}{a}(q)$

$J_\mu = \bar{\psi}\gamma_\mu\psi$. Therefore required matrix element becomes

$$\langle s\vec{q}|J_\mu(x)|r\vec{p}\rangle$$

$$= \langle 0| a^{(s)}(q)\, \bar{\psi}\gamma_\mu\psi(x)\, a^{(r)+}(p)|0\rangle$$

Next we move $a^{+(r)}$ to extreeme left and $\overset{(s)}{a}(q)$ to extreeme right using <u>anticommutators</u>

$$[a^{(r)}(p), a^{(s)+}(p')]_+ = \delta_{rs}\left(\frac{E_p}{M}\right)\delta(\vec{p}-\vec{p}')$$

and all other <u>anticommutators</u> are zero

$$[a^{r}(p)\ a^{(s)}(p')]_+ = [a^{+(r)}(p), a^{+(s)}(p')]_+ = 0$$

The anti commutators for the creation and annihilation operators lead to

$$[\psi(x), a^{+(r)}(p)]_+ = \delta_{rs} u^{(r)}(p) e^{-ipx}$$

$$[\bar{\psi}(x), a^{+(r)}(p)]_+ = 0.$$

Therefore we get

$$\langle 0 | a^{(s)}(q) \bar{\psi}(x) \gamma_\mu \psi(x) a^{+(r)}(p) | 0 \rangle$$

$$= u^{(r)}(p) e^{-ipx} - a^{+(r)}(p) \psi(x)$$

$$= \langle 0 | a^{(s)}(q) \bar{\psi}(x) \gamma_\mu (u^{(r)}(p) e^{-ipx} - a^{+(r)}(p) \psi(x)) | 0 \rangle$$

$$= \langle 0 | (\bar{u}^{(s)}(q) e^{iqx} - \bar{\psi}(x) a^{(s)}(q)) \gamma_\mu (u^{(r)}(p) e^{-ipx} - a^{+(r)}(p) \psi(x)) | 0 \rangle$$

$$= \bar{u}^{(s)}(q) \gamma_\mu u^{(r)}(p) e^{i(q-p)x}$$

$$+ \langle 0 | \bar{\psi}(x) a^{(s)}(q) \gamma_\mu u^{(r)}(p) e^{-ipx} | 0 \rangle$$ gives zero $a|0\rangle = 0$

$$+ \langle 0 | \bar{u}^{(s)}(q) e^{iqx} (-a^{+(r)}(p)) \psi(x) | 0 \rangle$$ becomes zero $\langle 0 | a^{+} = 0$

$$+ \langle 0 | \bar{\psi}(x) a^{(s)}(q) \gamma_\mu a^{(r)+}(p) \psi(x) | 0 \rangle$$

$$= \bar{u}^{(s)}(q) \gamma_\mu u^{(r)}(p) e^{i(q-p).x}$$

$$+ \langle 0 | \bar{\psi}(x) \left\{ \delta(\vec{q} - \vec{p}) \frac{E_p}{M} \delta_{rs} + a^{(r)+}(p) a^{(s)}(q) \right\} \gamma_\mu \psi(x) | 0 \rangle$$

The last term is seen to be zero because $\bar{\psi}(x)$ anticommutes with $a^{(r)\dagger}(p)$ and $\psi(x)$ anticommutes with $a^{(s)}(p)$. Therefore

last term

$$= \langle 0 |\, a^{(r)\dagger}(p) \cdot \bar{\psi}(x) \gamma_\mu \psi(x)\, a^{(s)}(q) | 0 \rangle = 0$$

$\therefore$ The desired matrix element becomes

$$\langle s \vec{q} \,|\, J_\mu \,|\, r \vec{p} \rangle$$

$$= \bar{u}^{(s)}(q) \gamma_\mu u^{(r)}(p)\, e^{i(q-p)x}$$

$$+ \frac{E_p}{M} \delta(\vec{q}-\vec{p}) \langle 0 | \bar{\psi}(x) \gamma_\mu \psi(x) | 0 \rangle$$

The second term is infinite (check this) and it will drop out if we take the current as normal product

$$J_\mu(x) = :\bar{\psi}(x) \gamma_\mu \psi(x):$$

In this case

$$\langle s \vec{q} \,|\, J_\mu(x) \,|\, r \vec{p} \rangle = \bar{u}^{(s)}(q) \gamma_\mu u^{(r)}(p)\, e^{i(q-p).x}$$

The manipulations can be streamlined by making use of a result (for normal product vacuum expectation values) similar to the Wick's theorem for time ordered product.