Express the field momentum

$$P^k = \int d^3x \big(\pi(x)\partial^k\phi(x) + \pi^*(x)\partial^k\phi(x)^*\big)$$

in terms of creation and annihilation operators.

DRAFT : To be checked for typographical errors

Comments This question essentially makes use of the orthogonality of the free particle wave functions. You should realise and use the fact that the free particle wave functions are orthogonal w.r.t. this scalar product. In fact those orthogonality relations were another part of question.

☺*Solution*: So let us write the expansion of the fields $\phi(x)$ and canonical momenta $\pi(x)$ in terms of creation and annihilation operators as

$$\begin{aligned}
\phi(x) &= \int \frac{d^3q}{2\omega_q}\Big(A(q)f_q(x) + B^*(q)f^*(q)\Big) &(1)\\
\partial_k\phi(x) &= \int \frac{d^3q}{2\omega_q}\Big((iq_k)A(q)f_q(x) + B^*(q)(-iq_k)f_q^*(x)\Big) &(2)\\
\partial_k\phi^*(x) &= \int \frac{d^3q}{2\omega_q}\Big((-iq_k)A^*(q)f_q*(x) + B(q)(iq_k)f_q(x)\Big) &(3)\\
\pi^*(x) &= \partial_0\phi(x) = \int \frac{d^3q}{2\omega_k}\Big(A(p)\partial_0 f_p(x) + B^*(p)\partial_0 f_p^*(x)\Big) &(4)\\
\pi(x) &= \partial_0\phi^*(x) = \int \frac{d^3q}{2\omega_p}\Big(A^*(p)\partial_0 f_p^*(x) + B(p)\partial_0 f_p(x)\Big) &(5)\\
& &(6)
\end{aligned}$$

The expression for the field momentum then becomes

$$\begin{aligned}
P^k &= \int d^3x\big(\pi(x)\partial^k\phi(x) + \pi^*(x)\partial^k\phi^*(x)\big) &(7)\\
&= \int d^3x\Big[\pi(x)\int\frac{d^3q}{2\omega_q}(iq^k)f_q(x)A(q) + \pi^*(x)\int\frac{d^3q}{2\omega_q}(-iq^k)f_q^*(x)B^*(q)\Big] + \text{two more terms}\\
&= \int\frac{d^3q}{2\omega_q}(iq^k)\Big\{A(q)\int d^3x f_q(x)\pi(x) - B^*(q)\int d^3x f_q^*(x)\pi^*(x)\Big\} + \text{two more terms}\\
&= \int\frac{d^3q}{2\omega_q}(q^k)i\Big\{A(q)\int d^3x f_q(x)\partial_0\phi^*(x) - B^*(q)\int d^3x f_q^*(x)\partial_0\phi(x)\Big\} + \text{two more terms}\\
&= -\int\frac{d^3q}{2\omega_q}(q^k)\int d^3x\Big\{A(q)(\phi(x), f_q(x)) + B^*(q)(f_q(x),\phi(x))\Big\} &(8)
\end{aligned}$$

Substituting the expressions for $\pi(x), \pi^*(x)$ we will get four terms having A^*A, B^*B, A^*B and AB^*. On integrating over$x$ the expression multiplying AB^* and A^*B for the field momentum vanish due to orthogonality of the plane wave solutions. In the other two terms having A^*A and B^*B, integration over x gives $2\omega_p \delta^3(p-q)$. The Dirac delta function can be used to do one of the two momentum integrals. Note that now we have delta function implying $\vec{p} = \vec{q}$ and therefore $\omega_p = \omega_q$. Thus we will be left with the desired final answer.