Prove that the free particle solutions $f_q(x)$ obey the orthononality relations

$$i \int d^3x f_q^*(x) \overleftrightarrow{\partial}_0 f_p(x) = 2\omega_q \delta(\vec{q} - \vec{p}).$$

and find the value of $\int d^3x f_q(x) \overleftrightarrow{\partial}_0 f_p(x)$.

[2](a) Lagrangian density for a charged free Klein Gordon field is

$$\mathcal{L}(x) = \partial_\mu \phi(x) \partial^\mu \phi^*(x) - m^2 \phi(x)\phi^*(x) \text{, where}$$

m = mass of the charged free Klein Gordon (spin = 0) particle.

[2](b) As known, free particle solutions of Klein-Gordon equation is

$$f_q(x) = \frac{1}{(2\pi)^{3/2}} e^{-iqx} \quad \text{—— (i)}$$

and $$f_q^*(x) = \frac{1}{(2\pi)^{3/2}} e^{iqx} \quad \text{—— (ii)}$$

$$\therefore \quad i\int d^3x\, f_q^*(x) \overleftrightarrow{\partial_0} f_p(x)$$

$$= \frac{i}{(2\pi)^3}\int d^3x\; e^{iqx} \overleftrightarrow{\partial_0} e^{-ipx}$$

$$= \frac{i}{(2\pi)^3}\int d^3x \left[e^{iqx} \frac{\partial}{\partial x_0} e^{-ipx} - \frac{\partial}{\partial x_0} e^{iqx}\, e^{-ipx}\right]$$

$$= \frac{i}{(2\pi)^3}\int d^3x \left[e^{iqx-ipx}(-i\omega_p) - (i\omega_q)\, e^{iqx-ipx}\right]$$

$$= -\frac{i^2}{(2\pi)^3}\int d^3x\; (\omega_p + \omega_q)\, e^{i(q-p)x} = 2\omega_q \delta(\vec{q}-\vec{p})$$

where $$\delta(\vec{q}-\vec{p}) = \frac{1}{(2\pi)^3}\int d^3x\, e^{i(q-p)x} \quad \text{—— (iii)}$$

Fig. 1

and the Klein-Gordon equation is

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = 0 \quad \text{———— (iv)}$$

$$\Rightarrow (\Box + m^2)\phi(x) = 0$$

$$\therefore i\int d^3x\, f_q^*(x) \overleftrightarrow{\partial_0} f_p(x) = 2\omega_q \delta(\vec{q} - \vec{p}) \text{ [Proved]}$$

now $\int d^4x\, f_q(x) \overleftrightarrow{\partial_0} f_p(x)$

$$= \int \frac{d^4x}{(2\pi)^3} \left\{ e^{-iqx} \overleftrightarrow{\partial_0} e^{-ipx} \right\}$$

$$= \int \frac{d^4x}{(2\pi)^3} \left\{ e^{-iqx} \frac{\partial e^{-ipx}}{\partial x^0} - \frac{\partial e^{-iqx}}{\partial x_0} e^{-ipx} \right\}$$

$$= \int \frac{d^4x}{(2\pi)^3} \left\{ (-ip) e^{-i(q+p)x} - (-iq) e^{-i(p+q)x} \right\}$$

$$= \int \frac{d^4x}{(2\pi)^3} -i \{\omega_p - \omega_q\} e^{-i(q+p)x}$$

$$= -i\int dt \int \frac{d^3x}{(2\pi)^3} (\omega_p - \omega_q) e^{-i(q+p)x}$$

A

$$= -i\int dt\, \delta(q+p)(\omega_p - \omega_q)$$

$$= \begin{cases} 0, & \text{if } q = -p \text{ as } \omega_p = \omega_{-p} \\ -i(\omega_p - \omega_q), & \text{if } q \neq -p \end{cases} \quad \text{(Ans)}$$

?

Fig. **2**

Comments

- Integrals are over space variables only; d^4x in the question was a typographical error and should be replaced with d^3x, a correction announced in the examination hall.
- In the 3rd line from bottom on page 2 of the scanned answer, $\delta(p+q)$ should be $\delta^3(p+q)$ and $\int dt$ should not be there.
- The last two lines of the scanned answer should be replaced by since $\vec{q} = \vec{p}$, we have $\omega_p = \omega_q$ Hence the integral is zero