Q3 Starting from the Lagrangian obtain the Hamiltonian for a free complex Klein Gordon field and show that

$$[H, \pi(x)] = -i(\nabla^2 - \mu^2)\phi^*(x)$$

The Lagrangian density for a charged scalar field is

$$\mathcal{L} = \partial_\mu \phi(x)\, \partial^\mu \phi(x) - \mu^2 \phi^*(x)\phi(x)$$

The canonical conjugate momenta are

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \partial_0 \phi^*$$

$$\pi^*(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*(x)} = \partial_0 \phi$$

∴ The Hamiltonian density is

$$\mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}$$

$$= \pi\pi^* + \pi^*\pi - (\partial_0\phi^*\partial^0\phi - (\nabla\phi)^*(\nabla\phi) - m^2\phi^*\phi)$$

$$= 2\pi^*\pi - (\pi^*\pi - (\nabla\phi)^*(\nabla\phi) - m^2\phi^*\phi)$$

$$= \pi^*\pi + (\nabla\phi)^*(\nabla\phi) + m^2\phi^*\phi$$

The Hamiltonian is given by

$$H = \int \mathcal{H}\, d^4x$$

$$= \int (\pi^*(x)\pi(x) + (\nabla\phi)^*\nabla\phi + m^2\phi^*\phi)\, d^4x$$

The equal time commutation relations are

$$[\phi(\vec{x},t)\ \ \pi(\vec{x}',t)] = i\delta(\vec{x}-\vec{x}')$$

$$[\phi^*(\vec{x}',t)\ \ \pi(\vec{x},t)] = i\delta(\vec{x}-\vec{x}')$$

We need to compute $[H, \pi(x)]$. Since H is constant of motion, we can take H at same time as t and use ETCR

$$H = \int d^3y \left(\pi^*(\vec{y},t)\,\pi(\vec{y},t) + (\nabla\phi)^*(\vec{y},t)\,\nabla\phi(\vec{y},t) + m^2\phi^*(\vec{y},t)\,\phi(y,t) \right)$$

In $[H, \pi(\vec{x},t)]$, $\pi(\vec{x},t)$ commutes with all operators except $\phi^*(x,t)$. Therefore

$$[\pi^*(\vec{y},t)\,\pi(\vec{y},t),\ \pi(\vec{x},t)] = 0$$

$$[\nabla_y\phi^*(\vec{y},t)\,\nabla\phi(\vec{y},t),\ \pi(x,t)]$$

$$= [\nabla_y\phi^*(\vec{y},t),\ \pi(x,t)]\,\nabla_y\phi(\vec{y},t)$$

$$= \left(\nabla_y\,[\phi^*(\vec{y},t),\ \pi(x,t)]\right)\nabla_y\phi(\vec{y},t)$$

$$= i\nabla_y\,\delta^{(3)}(\bar{x}-\bar{y})\,\nabla\phi(\vec{y},t)$$

Also $[\phi^*(\vec{y},t)\,\phi(\vec{y},t),\ \pi(\vec{x},t)]$

$$= [\phi^*(\vec{y},t),\ \pi(\vec{x},t)]\,\phi(\vec{y},t)$$

$$= i\delta^{(3)}(\vec{x}-\vec{y})\,\phi(\vec{y},t)$$

Putting the last three commutators together we get

$$[H, \pi(\vec{x},t)]$$

$$= \int d^3y \left[\pi^*(y,t)\,\pi(\vec{y},t) + \nabla_y \phi^*(y,t)\,\nabla_y \phi(y,t) + m^2 \phi^*(y,t)\,\phi(y,t),\ \pi(\vec{x},t) \right]$$

$$= \int d^3y \left(0 + i \nabla_y \delta^{(3)}(\vec{x}-\vec{y})\ \nabla_y \phi(y,t) + i m^2 \delta(\vec{x}-\vec{y})\,\phi(\vec{y},t) \right)$$

Use Dirac δ functions to carry out y integral

$$= -i\, \nabla^2 \phi(\vec{x},t) + i m^2 \phi(\vec{x},t)$$

Note $\int \nabla_y \delta^3(\vec{x}-\vec{y})\, \nabla_y \phi(\vec{y},t)\, dy = (-)\, \nabla_y^2 \phi(\vec{y},t)\big|_{\vec{y}=\vec{x}} = -\nabla^2 \phi(\vec{x},t)$

$$= -i\,(\nabla^2 - m^2)\,\phi(\vec{x},t)$$

$$\therefore\ [H, \pi(\vec{x},t)] = -i\,(\nabla^2 - m^2)\,\phi(\vec{x},t).$$