QFT-15 Solved Problem
Pion Life Time

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Question

The interaction Hamiltonian for pion decay $\pi^- \longrightarrow e^- + \bar{\nu}$ can be written as

$$\mathscr{H}_{\rm int} = \frac{g}{\sqrt{2}}\bar{\psi}_e(x)\gamma_\mu(1-\gamma_5)\psi(x)_\nu\partial^\mu\phi^-_\pi(x) + h.c.$$

Show that the decay rate is given by

$$\Gamma = \frac{g^2}{8\pi}\frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\pi^3}.$$

Solution

The structure of S matrix element Let Q, p, k denote the four momenta of the pion the electron and the neutrino. Then momentum conservation requires

$$Q = p + k.$$

The S matrix element has the form

$$S_{fi} = \delta_{fi} - i(2\pi)^4\delta^{(4)}(P - p - k)T_{fi}$$

where

$$T_{fi} = \frac{1}{(2\pi)^{9/2}}\sqrt{\frac{1}{2\omega_Q}\frac{m_p}{E_p}\frac{1}{2E_k}}\ \ m_{fi}$$

and m_{fi} is a Lorentz invariant function of momenta. We make use of Dirac equation to simplify the matrix element m_{fi}.

$$m_{fi} = \frac{g}{\sqrt{2}}\bar{u}^{(r)}(p)(\gamma_\mu iQ_\mu)(1-\gamma_5)v^{(s)}(k) \quad (1)$$

$$= \frac{1}{(2\pi)^{9/2}}\frac{g}{\sqrt{2}}\bar{u}^{(r)}(p)i(\not{p} + \not{k})(1-\gamma_5)v^{(s)}(k) \quad (2)$$

$$= \frac{m_e}{(2\pi)^{9/2}}\frac{ig}{\sqrt{2}}\bar{u}^{(r)}(p)(1-\gamma_5)v^{(s)}(k) \quad (3)$$

where use has been made of $Q = p + k$, the fact that neutrino has zero mass and therefore $\bar{u}(p)(\not{p} - m) = 0$ for the electron. For the neutrino we have used $\not{k} v(k) = 0$.

Transition Probability per unit time per unit volume
The expression

$$-i(2\pi)^4 \delta^{(4)}(P - p - k) T_{fi}$$

gives the transition *amplitude* for the process and the square will be the probability for the transition to take place.

Squaring the transition matrix element and dropping one factor of $(2\pi)^4 \delta^{(4)}(Q - p - k)$ will give the transition probability per unit time per unit volume. Denoting it by W_{fi} we have

$$W_{fi} = (2\pi)^4 \delta^{(4)}(P - p - k) \frac{1}{(2\pi)^9} \Big\{ \frac{1}{2\omega_Q} \frac{m_e}{E_e} \frac{1}{2E_k} \Big\} \times |m_{fi}|^2. \tag{4}$$

Since we are not interested transition probability *per unit volume*, but in decay probability per unit time for a single particle, we must divide by the number of particles per unit volume, *i.e.* the particle number density. The density ρ is simply the same quantity that appears in the equation of continuity.

$$w_{fi} = (2\pi)^4 \delta^{(4)}(Q - p - k) \frac{1}{\rho_i} \times \frac{1}{(2\pi)^9} \Big\{ \frac{1}{2\omega_Q} \frac{m_e}{E_e} \frac{1}{2E_k} \Big\} |m_{fi}|^2 \tag{5}$$

$$\text{where the density } \rho_i \text{ is } \frac{1}{(2\pi)^3}. \tag{6}$$

$$= \delta^{(4)}(Q - p - k) \Big(\frac{1}{8\pi^2 m_\pi} \Big) \Big\{ \frac{m_e}{E_e} \frac{1}{2E_k} \Big\} |m_{fi}|^2 \tag{7}$$

Averaging over initial and summing over final states Since the decaying particle does not carry spin, it has only one state and no averaging over the initial state is required. The sum over final states is carried out by

$$\text{sum over all final states} \longrightarrow \text{(sum over final momenta) and (sum over final spins)} \tag{8}$$

$$\text{sum over final momenta} \longrightarrow \iint d^3p \; d^3k. \tag{9}$$

Putting everything together Collecting all the above factors, the expression for the pion decay rate (=transition probability per unit time) becomes

$$\Gamma = \frac{1}{8\pi^2} \times \frac{m_e}{m_\pi E_p E_k} \iint d^3p \; d^3k \delta^{(4)}(Q - p - k) \; |m_{fi}|^2. \tag{10}$$

Carrying out momentum integrals We work in the rest frame of the pion and let m_π denote mass of the pion.. Then $\vec{Q} = 0, Q^0 = m_\pi, \vec{p} + \vec{k} = 0$, Since

$$\delta^{(4)}(Q - p - k) = \delta^{(3)}(\vec{p} + \vec{k}) \delta(m_\pi - E_e - E_\nu)$$

Here E_e, E_ν denote the energies of the electron and the neutrino. The first delta function, $\delta^{(3)}(\vec{p}+\vec{k})$, can be used to do the $\vec{k}$ integrals. This sets $\vec{k}=-\vec{p},\ E_k=|\vec{p}|$, we get

$$
\begin{aligned}
I &= \int d^3p\delta(m_\pi - E_e - E_\nu). &(11)\\
&= \int 4\pi|\vec{p}|^2\, d|\vec{p}|\ \delta\Big(m_\pi - \sqrt{|p|^2+m^2} - k\Big) &(12)\\
&= 4\pi|\vec{p}|^2\Big(\frac{|\vec{p}|}{E_e}+1\Big)^{-1} = 4\pi|\vec{p}|^2\frac{E_e}{p+E_e} = 4\pi|\vec{p}|^2\frac{E_e}{m_\pi}. &(13)
\end{aligned}
$$

Here $|\vec{p}|$ is the momentum of the electron in the rest frame and is obtained by solving the energy conservation equation:

$$|\vec{p}| + \sqrt{|\vec{p}|^2+m_e^2} = m_\pi \implies |\vec{p}| = \frac{m_\pi^2 - m_e^2}{2m_\pi}. \tag{14}$$

Squaring and summing over final spins Squaring and summing over all spins we get

$$
\begin{aligned}
|m_{fi}|^2 &= \frac{m_e^2 g^2}{2}\sum_{r,s}\bar{u}(p)(1-\gamma_5)v^{(s)}(k)\times\bar{v}(k)(1-\gamma_5) &(15)\\
&= \frac{m_e^2 g^2}{2}\sum_{r,s}\text{trace}\big\{u^{(r)}(p)\bar{u}(p)(1-\gamma_5)v^{(s)}(k)\bar{v}(k)(1-\gamma_5)\big\} &(16)
\end{aligned}
$$

where trace is taken over Dirac indices. We now use the identities

$$
\begin{aligned}
\sum_r u^{(r)}(p)_\alpha\bar{u}^{(s)}(p)_\beta &= \Big(\frac{\not{p}+m_e}{m_e}\Big)_{\alpha\beta} \qquad \text{for electrons} &(17)\\
\sum_s u^{(s)}(k)_\alpha\bar{u}^{(s)}(k)_\beta &= \not{k}_{\alpha\beta} \qquad \text{for neutrinos.} &(18)
\end{aligned}
$$

The sum over final particle spins will be done and simplified.

$$
\begin{aligned}
\sum_{r,s}|m_{fi}|^2 &= \Big(\frac{g^2}{2}m_e^2\Big)\text{trace}\Big\{\frac{\not{p}+m_e}{2m_e}(1-\gamma_5)\not{k}(1-\gamma_5)\Big\} &(19)\\
&= \frac{g^2m_e^2}{2}\frac{1}{2m_e}\times\text{trace}\big\{(\not{p}+m_e)(-\not{k})(1-\gamma_5)(1-\gamma_5)\big\} &(20)\\
&= \frac{g^2m_e}{4}\times\text{trace}\big\{(\not{p}+m_e)(-\not{k})2(1-\gamma_5)\big\} &(21)\\
&= g^2m_e(2p\cdot k) &(22)
\end{aligned}
$$

Now calculate $p\cdot k$ using energy momentum conservation

$$2p\cdot k = (p+k)^2 - p^2 - k^2 = Q^{02} - m_e^2 = m_\pi^2 - m_e^2. \tag{23}$$

Therefore

$$|m_{fi}|^2 = (g^2m_e)(m_\pi^2 - m_e^2). \tag{24}$$

Therefore, putting (10), (13) and (24) together, the final expression for the decay width becomes

$$\begin{aligned}\Gamma &= \frac{g^2}{2}\frac{1}{(4\pi^2)}\left\{\frac{1}{2m_\pi}\frac{1}{2E_\nu}\frac{m_e}{E_e}\right\}\iint d^3p\, d^3p\ \delta^{(4)}(Q-p-k)|m_{fi}|^2 && (25)\\ &= \frac{g^2}{8\pi^2}\left\{\frac{1}{2m_\pi}\frac{1}{2|\vec{p}|}\frac{m_e}{E_e}\right\}\left\{\frac{4\pi|p|^2E_e}{m_\pi}\right\}\{2m_e(m_\pi^2-m_e^2)\} && (26)\\ &= \frac{g^2}{4\pi}\frac{m_e^2}{m_\pi^2}|\vec{p}|\,(m_\pi^2-m_e^2), \qquad \because E_\nu = |\vec{p}|. && (27)\end{aligned}$$

Note that neutrino is massless and hence using $|\vec{p}| = |\vec{k}| = E_\nu$ along with Eq.(14) we get

$$\Gamma = \frac{g^2}{8\pi}\frac{m_e^2(m_\pi^2-m_e^2)^2}{m_\pi^3}. \qquad (28)$$