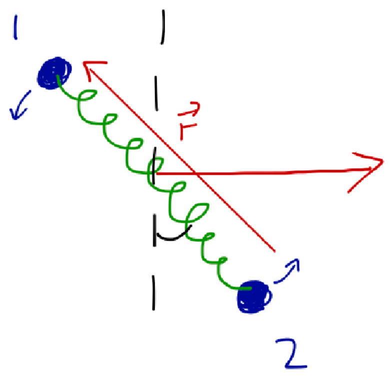
UNIVERSITY OF HYDERABAD
SCHOOL OF PHYSICS

M.Sc.-I/IMSc.-IV Classical Mechanics **June 30, 2018**
May 14-Jul 6 (2018) **MM:**

TEST-I

1 Question

Suppose we connect two bodies of masses $m_1 = m_2 = m$ by a spring with constant k and natural length ℓ, and then allow them to slide across a horizontal frictionless table.



Two bodies connected by a spring

(a) Setup the Lagrangian for the system in Cartesian coordinates $\vec{r}_1, \vec{r}_2$. [4]

(b) Is this an example of central force force problem? [2]

(c) Define the center of mass $\vec{R}$ and relative coordinate $\vec{r}$ [4]

$$\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}, \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

and express the Lagrangian in terms of

(i) new coordinates $\vec{r}$ and $\vec{R}$

(ii) polar coordinates r, ϕ defined by $x = r\cos\phi, y = r\sin\phi$ for relative motion.

(d) Find cyclic coordinates and conserved quantities. [5]

(e) Find the effective potential for the radial motion and plot it for (i)zero angular momentum, and (ii) non zero angular momentum. [2+3]

2 Solution

(a) The kinetic energy is given by

$$T = \frac{m}{2}\left(\dot{\vec{r}}_1^2 + \dot{\vec{r}}_2^2\right)$$

Let k denote the spring constant the spring. Then the potential energy is

$$V = \frac{1}{2}k(|\vec{r}_1 - \vec{r}_2| - \ell)^2$$

The Lagrangian is therefore given by

$$L = T - V = \frac{m}{2}\left(\dot{\vec{r}}_1^2 + \dot{\vec{r}}_2^2\right) - \frac{1}{2}k(|\vec{r}_1 - \vec{r}_2| - \ell)^2$$

(b) Yes, because the interaction depends only on the distance between the two bodies.

(c) The two coordinates $\vec{r}_1, \vec{r}_2$ can be written in terms of relative and center of mass coordinates.

$$\vec{r}_1 = \vec{R} + \frac{1}{2}\vec{r}, \qquad \vec{r}_2 = \vec{R} - \frac{1}{2}\vec{r}$$

Hence the kinetic energy is

$$\begin{aligned} T &= \frac{m}{2}\left(|\dot{\vec{R}} + \frac{1}{2}\dot{\vec{r}}|^2 + |\dot{\vec{R}} - \frac{1}{2}\dot{\vec{r}})|^2 & (1) \\ &= m\dot{\vec{R}}^2 + \frac{m}{4}\dot{\vec{r}}^2 & (2) \\ &= \frac{1}{2}\left\{M\dot{\vec{R}}^2 + \mu\dot{\vec{r}}^2\right\} & (3) \end{aligned}$$

Here $M = 2m, \mu = m/2$ are total mass and reduced mass respectively. The potential energy becomes

$$V = \frac{k}{2}(r - \ell)^2$$

In polar coordinates the Lagrangian Takes the form

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 - \frac{k}{2}(r - \ell)^2$$

(d) The cyclic coordinates are $\vec{R}$ and θ. The conserved quantities are total momentum of the system, $M\dot{\vec{R}}$, and the angular momentum $p_\theta = \mu r^2\dot{\theta}$.

(e) The total energy is given by

$$\begin{aligned} E &= \frac{\mu}{2}\dot{r}^2 + \frac{\mu}{2}r^2\dot{\theta}^2 + \frac{1}{2}(r - \ell)^2 & (4) \\ &= \frac{\mu}{2}\dot{r}^2 + \frac{p_\theta^2}{2\mu r^2} + \frac{k}{2}(r - \ell)^2 & (5) \\ & & (6) \end{aligned}$$

Hence the effective potential for radial motion is given by

$$V_{\text{eff}} = \frac{k}{2}(r-\ell)^2 + \frac{p_\theta^2}{2\mu r^2} = \frac{k}{2}\{(r-\ell)^2 + \frac{\lambda}{r^2}\}$$

where $\lambda = \frac{p_\theta^2}{k\mu}$. A few plots for $\ell = 1$ and for $\lambda = 0, 0.2$ and 2,

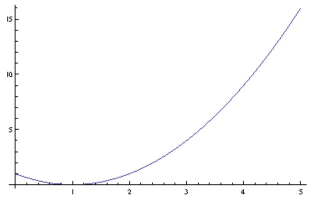


Fig. **1** $p_\theta = 0$

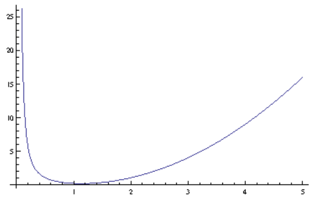


Fig. **2** $p_\theta = 0.2$

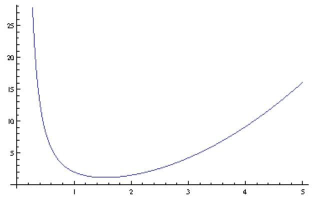


Fig. **3** $p_\theta = 2$