University of Hyderabad
School of Physics

M.Sc.-I/IMSc.-IV Classical Mechanics **June 15, 2018**

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Quiz-II-Solutions

[1] Obtain the Lagrangian for a pendulum attached to spring as shown in the figure.

[2] Find equilibrium points and Lagrangian in small amplitude approximation.[2]

[3] Write the equations of motion from the Lagrangian in small amplitude approximation.The natural length of the spring is a and k is the spring constant.

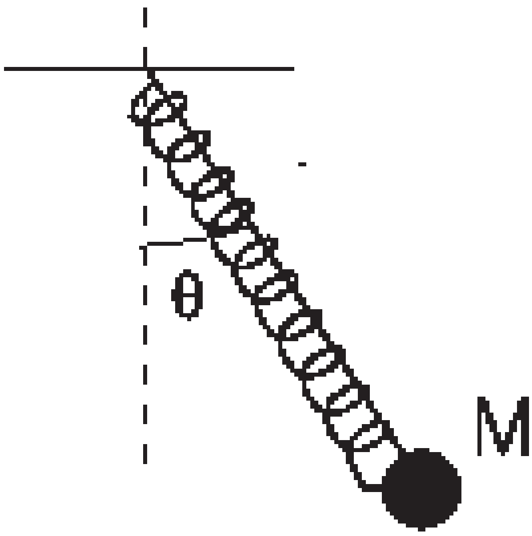

Fig. **1** Pendulum With Spring

☺*Solution*:

[1] Let r, θ denote the polar coordinates of the bob at time t. We choose the vertical plane of oscillations as $X - Y$ plane with Y axis pointing upwards. Then the Cartesian coordinates of the bob are given by

$$x = r\cos\theta, \qquad y = -r\sin\theta. \tag{1}$$

This gives

$$\dot{x} = \dot{r}\,\cos\theta - r\sin\theta\,\dot{\theta}, \qquad \dot{y} = -\dot{r}\,\sin\theta. - r\cos\theta\,\dot{\theta} \tag{2}$$

Hence kinetic energy is given by

$$\begin{aligned}
\text{K.E.} &= \frac{1}{2}m(\dot{(x)}^2 + \dot{y}^2) && (3)\\
&= \frac{1}{2}m\big((\dot{r}\,\cos\theta - r\sin\theta\,\dot{\theta})^2 + (-\dot{r}\,\sin\theta. - r\cos\theta\,\dot{\theta})^2\big) && (4)\\
&= \frac{1}{2}m(\dot{r}^2 + mr^2\dot{\theta}^2) && (5)
\end{aligned}$$

The potential energy is given by

$$\begin{aligned}
\text{P.E.} &= mgy + \frac{k}{2}(r-a)^2 && (6)\\
V(r,\theta) &= -mgr\cos\theta + \frac{k}{2}(r-a)^2 && (7)
\end{aligned}$$

The Lagrangian is therefore given by

$$L = \frac{1}{2}m(\dot{r}^2 + mr^2\dot{\theta}^2) + mgr\cos\theta - \frac{k}{2}(r-a)^2 \tag{8}$$

[2] The equilibrium values of r, θ are given by

$$\begin{aligned}
\frac{\partial V(r,\theta)}{\partial\theta} = 0 \quad &\Longrightarrow \quad \sin\theta = 0, && (9)\\
\frac{\partial V(r,\theta)}{\partial r} = 0 \quad &\Longrightarrow \quad -mg\cos\theta + k(r-a) = 0 && (10)
\end{aligned}$$

Eq.(9) gives $\theta = 0$ and (10) implies $r = a + mg/k$.
Define displacement from the equilibrium value $u = r - (a + mg/k)$. Denote the equilibrium value of r by r_0, substituting $r = u + r_0$.

$$\begin{aligned}
L &= \frac{1}{2}m\big(\dot{u}^2 + (u+r_0)^2\dot{\theta}^2\big) - mg(u+r_0)\cos\theta - \frac{k}{2}(r+mg/k)^2 && (11)\\
&= \frac{1}{2}m\big(\dot{u}^2 + (u+r_0)^2\dot{\theta}^2\big) - mg(u+r_0)(1 - \frac{\theta^2}{2} + \cdots) - \frac{k}{2}u^2 - \frac{1}{2}(mg/k)^2. && (12)
\end{aligned}$$

Retaining only quadratic terms in $\dot{u}, \dot{\theta}, u$ and θ we get the Lagrangian for small amplitude approximation as

$$L = \frac{1}{2}m\Big(\dot{u}^2 + r_0^2\dot{\theta}^2\Big) - \frac{1}{2}mgr_0\theta^2 - \frac{k}{2}u^2 - \frac{1}{2}(mg/k)^2 - \frac{1}{2}mgr_0\theta^2 \tag{13}$$

[3] The equation of motion for small amplitudes is now obtained easily as

$$\begin{aligned}
\frac{\partial L}{\partial\dot{u}} = m\ddot{u}, \quad \frac{\partial L}{\partial u} = -ku \quad &\Rightarrow \quad \ddot{u} = -\frac{k}{m}u, && (14)\\
\frac{\partial L}{\partial\dot{\theta}} = mr_0^2\ddot{\theta}, \quad \frac{\partial L}{\partial\theta} = -mgr_0\theta \quad &\Rightarrow \quad \ddot{\theta} = -\frac{g}{r_0}u. && (15)
\end{aligned}$$