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P1/Q14/§§7.6

§§7.6 Q14 For $-1 < p < 1$ show that

$$\int_0^\infty \frac{x^p\,dx}{x^2+2x\cos\alpha+1} = \pi \sin p\alpha \operatorname{cosec} p\pi \operatorname{cosec}\alpha$$

Let $f(z) = z^p/(z^2+2z\cos\alpha+1)$

where z^p is defined by

$$z^p = r^p e^{i\theta p} \quad 0 < \theta < 2\pi$$

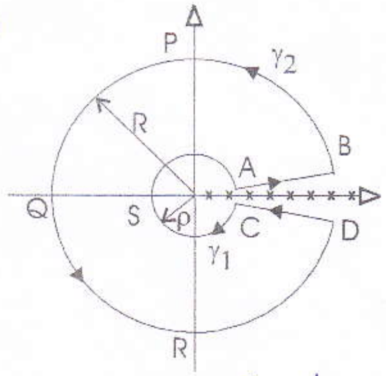

Fig. 1 Contour Γ

Therefore $f(z)$ has a branch cut running along $\theta = 0$. We set up the integral of $f(z)$ around closed contour Γ of Fig 1. In the limit $\rho \to 0$, $R \to \infty$, the integrals along circular arcs vanish and we have

$$\lim \oint_\Gamma f(z)\,dz = \lim \int_{AB} f(z)\,dz + \lim \int_{BPQRD} f(z)\,dz + \int_{DC} f(z)\,dz$$

$$+ \lim \int_{CSA} f(z)\,dz$$

$$= \lim_{\substack{\rho\to 0\\ R\to\infty}} \left(\int_{AB} f(z)\,dz - \int_{CD} f(z)\,dz \right) \qquad (1)$$

We parametrize the two straight line segments AB, CD as

$AB: z = re^{i\epsilon}, \rho < r < R$ $\qquad$ $CD: z = re^{(2\pi-\epsilon)i}$ $\rho < r < R$

$dz = dr e^{i\epsilon}$ $\qquad$ $dz = dr e^{(2\pi-\epsilon)i}$

$$f(z) = \frac{r^p e^{i\epsilon p}}{z^2+2z\cos\alpha+1} \qquad f(z) = \frac{r^p e^{(2\pi-\epsilon)ip}}{z^2+2z\cos\alpha+1} \qquad (2)$$

Using (2) in Eq (1) we get

$$\lim_{\substack{\rho\to 0\\ R\to\infty}} \oint_{\Gamma} f(z)\,dz = \int_0^{\infty} \frac{r^p}{r^2+2r\cos\alpha+1}\,(1-e^{i2\pi(p+1)})\,dr$$

$$= (1-e^{2\pi i(p+1)})\int_0^{\infty}\frac{r^p\,dr}{r^2+2r\cos\alpha+1}$$

$$\therefore \int_0^{\infty}\frac{x^p\,dx}{x^2+2x\cos\alpha+1} = (1-e^{2\pi i(p+1)})^{-1}\oint_{\Gamma} f(z)\,dz \quad \text{---- (3)}$$

$f(z)$ has simple poles at points where $z^2+2z\cos\alpha+1=0$

or $z = \dfrac{-2\cos\alpha \pm \sqrt{4\cos^2\alpha-4}}{2}$

$= -\cos\alpha \pm i\sin\alpha = -e^{\pm i\alpha}$

$\therefore z^2+2z\cos\alpha+1 = (z+e^{i\alpha})(z+e^{-i\alpha})$

Hence

$$\operatorname{Res}\left\{\frac{f(z)}{z^2+2z\cos\alpha+1}\right\}_{z=-e^{i\alpha}} \qquad z = e^{i\alpha}e^{i\pi}$$

$$= e^{-ip\pi}\,\frac{e^{i\alpha p}\,(z+e^{i\alpha})}{z^2+2z\cos\alpha+1}\Bigg| = \left(\frac{e^{i\alpha p}}{z+e^{-i\alpha}}\right)\Bigg| = e^{i\pi p}e^{i\alpha p}$$

---- (4)

Similarly

$$\operatorname{Res}\left\{\frac{f(z)}{z^2+2z\cos\alpha+1}\right\}_{z=-e^{-i\alpha}} = \left(\frac{e^{-i\alpha p}}{z+e^{+i\alpha}}\right)\Bigg|_{z=-e^{-i\alpha}}$$

$\therefore$ Sum of residues

$$= \left(\frac{e^{i\alpha p}}{-e^{i\alpha} + e^{-i\alpha}} + \frac{e^{-i\alpha p}}{-e^{-i\alpha} + e^{i\alpha}} \right) e^{ip\pi}$$

$$= e^{ip\pi} \frac{(-e^{i\alpha p} + e^{-i\alpha p})}{e^{i\alpha} - e^{-i\alpha}} = e^{ip\pi} \frac{-2i \sin \alpha p}{2i \sin \alpha}$$

$$\therefore \int_0^\infty \frac{x^p \, dx}{x^2 + 2x\cos\alpha + 1} = 2\pi i \times e^{ip\pi} (-1) \sin \alpha p \operatorname{cosec} \alpha$$

$$\times \frac{1}{1 - e^{2\pi i p}}$$

$$= (-1) \frac{2\pi i \, e^{ip\pi} \sin \alpha p \operatorname{cosec} \alpha}{e^{i\pi p} (-2i) \sin \pi p}.$$

$$= \pi \sin p\alpha \operatorname{cosec} p\pi \operatorname{cosec} \alpha.$$