P1/Q12/§§7.6

§§7.6 Q12 Compute $\int_1^\infty \frac{(x-1)^{p-1}}{x^3}\,dx$ using complex contour integration. $(0<p<3)$.

The given integral can be written as

$$\int_1^\infty \frac{(x-1)^{p-1}}{x^3}\,dx = \int_0^\infty \frac{x^{p-1}}{(x+1)^3}\,dx \quad \ldots (1)$$

We assume the definition

$$z^{p-1} = r^{p-1} e^{i\theta(p-1)}, \quad 0<\theta<2\pi$$

and set up $\oint f(z)\,dz$ along closed contour Γ of Fig 1 with

$$f(z) = \frac{z^{p-1}}{(z+1)^3}.$$

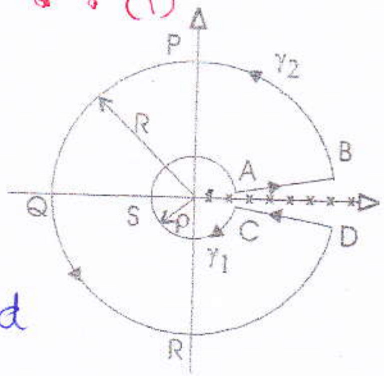

Fig. 1 Contour Γ

Then

$$\oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BPQRD} f(z)\,dz + \int_{DC} f(z)\,dz + \int f(z)\,dz$$

In the limit $\rho\to 0, R\to\infty$, the CSA integrals along circular arcs BPQRD and CSA vanish. Hence

$$\oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz + \int_{DC} f(z)\,dz = \int_{AB} f(z)\,dz - \int_{CD} f(z)\,dz \quad \ldots (2)$$

Next, to set up the integrals along AB and CD we use

AB: $z = re^{i\epsilon}$

$z^{p-1} = r^{p-1} e^{i\epsilon(p-1)}$

$dz = dr\, e^{i\epsilon}$

CD: $z = re^{i(2\pi-\epsilon)}, \quad \rho<r<R$

$dz = dr\, e^{i(2\pi-\epsilon)}$

$z^{p-1} = r^{p-1} e^{i(2\pi-\epsilon)(p-1)}$

P2/Q12/§§7.6

In the limit $\epsilon \to 0$ we get

$$\lim \int_{AB} f(z)\,dz = \int_0^\infty \frac{\ \ ^{p-1}}{(\ \ +1)^3}\,d\gamma \quad \text{and}$$

$$\int_{CD} f(z)\,dz = \int_0^\infty \frac{\gamma^{p-1}}{(\gamma+1)^3}\, e^{2\pi i p}\, d\gamma$$

|| lim $\rho \to 0$, $R \to \infty$ is understood

Therefore, from (2)

$$\oint_\Gamma f(z)\,dz = (1-e^{2\pi i p}) \int_0^\infty \frac{\gamma^{p-1}}{(\gamma+1)^3}\,d\gamma$$

$$= -2ie^{i\pi p} \sin \pi p \int_0^\infty \frac{\gamma^{p-1}}{(\gamma+1)^3}\,d\gamma \quad \ldots \ (3)$$

The integral along closed contour Γ can now be evaluated using residue theorem. The contour Γ encloses a pole or order 3 at $z=-1$. Therefore we compute

$$\operatorname{Res}\{f(z)\}_{z=-1} = \frac{d^2}{dz^2}(z+1)^3 f(z)\Big|_{z=-1}$$

$$= \frac{d^2}{dz^2} z^p \Big|_{z=e^{i\pi p}}.$$

$$= p(p-1)\,\gamma^p e^{i\pi p} \quad \ldots \ (4)$$

Using (4) in L.H.S. of (3) gives

$$2\pi i\, p(p-1)\, e^{i\pi p} = (-2i) e^{i\pi p} \sin(p\pi) \int_1^\infty \frac{\gamma^p}{(\gamma+1)}\,d\gamma$$

$$\therefore \int_0^\infty \frac{(x-1)^{p-1}}{x^3}\,dx = \int_1^\infty \frac{\gamma^{p-1}}{(\gamma+1)^3}\,d\gamma$$

$$= p(1-p)\,\pi / \sin(\pi p).$$