13-8-2017 $\qquad$ p1/Q11/§87.6

§87.6 Q11 Compute the integral $\int_1^\infty \frac{(x-1)^{p-1}}{x^2}\,dx$ by the method of contour integration.

The given integral can be written as

$$\int_1^\infty \frac{(x-1)^{p-1}}{x^2}\,dx = \int_0^\infty \frac{x^{p-1}}{(x+1)^2}\,dx$$

Next we set up integral $\oint_\Gamma f(z)\,dz$ where $f(z) = \frac{z^{p-1}}{(z+1)^2}$ along the contour Γ of Fig 1. Then

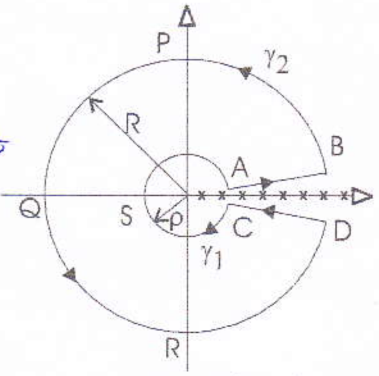

Fig. 1 Contour Γ.

$$\oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BPQRD} f(z)\,dz + \int_{DC} f(z)\,dz + \int_{CSA} f(z)\,dz \quad \text{--- (1)}$$

The integrals along circular arcs BPQRD and CSA go to zero in the limit $R\to\infty$ and $\rho\to 0$, respectively. Therefore,

$$\oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz + \int_{DC} f(z)\,dz$$

$$= \int_{AB} f(z)\,dz - \int_{CD} f(z)\,dz \quad \ldots (2)$$

Next to set up integrals of $f(z)$ along st lines AB and CD we use

AB: $z = re^{i\epsilon}$, $0<r<R$ $\qquad$ CD: $z = re^{i(2\pi-\epsilon)}$, $0<r<R$

$dz = dr\,e^{i\epsilon}$ $\qquad$ $dz = dr\,e^{i(2\pi-\epsilon)}$

$z^p = r^p e^{i\epsilon p}$ $\qquad$ $z^p = r^p e^{i(2\pi p-\epsilon p)}$

In lim $\epsilon\to 0$, $f(z) = \frac{r^{p-1}}{(r+1)^2}$ and $f(z) = \frac{r^{p-1}e^{i2\pi(p-1)}}{(r+1)^2}$

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$$\therefore \oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz - \int_{CD} f(z)\,dz$$

$\lim \epsilon \to 0, \rho \to 0$, $R \to \infty$ are understood here

$$= \int_0^\infty \frac{r^{p-1}(1-e^{2\pi i p})}{(r+1)^2}\,dr$$

$$= e^{i\pi p}(-2i)\sin(\pi p)\int_0^\infty \frac{r^{p-1}}{(r+1)^2}\,dr \quad \ldots (3)$$

The complex contour integration of $f(z)$ along Γ can be computed using residue theorem. The contour encloses a double pole at $z=-1$.

$$\text{Res}\left\{\frac{z^{p-1}}{(z+1)^2}\right\} = \frac{1}{2}\frac{d}{dz}\left\{\frac{(z+1)^2 z^{p-1}}{(z+1)^2}\right\}\Bigg|_{z=-1}$$

$$= \frac{1}{2}(p-1)z^{p-1}\Big|_{z=e^{i\pi}}$$

$$= \frac{1}{2}(p-1)e^{ip\pi - ip} = (1-p)e^{ip\pi}\times\frac{1}{2} \quad \ldots (4)$$

Using this we get

$$\oint f(z)\,dz = 2\pi i(1-p)e^{ip\pi} \quad \ldots (5)$$

The answer (5) substituted in (3) gives.

$$\frac{1}{2}(-2i)(\sin\pi p)e^{i\pi p}\int_0^\infty \frac{r^{p-1}}{(r+1)^2}\,dr = 2\pi i(p-1)e^{ip\pi}$$

Hence

$$\int_1^\infty \frac{(x-1)^{p-1}}{x^2}\,dx = \int_0^\infty \frac{r^{p-1}}{(r+1)^2}\,dr = \frac{p(1-p)\pi}{2\sin p\pi}$$

$$= \frac{1}{2}p(1-p)\,\text{cosec}(p\pi)$$