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§§7.6 Q9 Integrate $\int_0^\infty \frac{x^{p-1}}{1+x^a}\,dx$, $0<p<a$ using the method of contour integration.

We take the definition

$$z^a = r^a \exp(i\theta a), \quad 0<\theta<2\pi$$

so that the branch cut for z^a is along $\theta = 0$. Integrate

$f(z) = \frac{z^{p-1}}{1+z^a}$ along the closed contour Γ of figure 1. Then

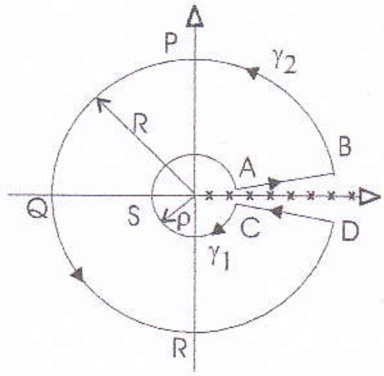

Fig. 1 Contour Γ

$$\oint_\Gamma f(z)dz = \int_{AB} f(z) + \int_{BPQRD} f(z)dz + \int_{DC} f(z)dz + \int_{CSA} f(z)dz \quad \text{--- (1)}$$

Under the given condition $0<p<a$, the integrals along circular arcs CSA and BPQRD vanish in the limits $\rho\to 0$ and $R\to\infty$ respectively. Hence

$$\lim \oint_\Gamma f(z)dz = \int_{AB} f(z)dz + \int_{DC} f(z)dz.$$

$$= \int_{AB} f(z)dz - \int_{CD} f(z)dz. \quad \text{---- (2)}$$

Along the straight lines AB and CD we have

AB: $z = re^{i\epsilon}$ $\rho<r<R$

$dz = dr\,e^{i\epsilon}$

$$f(z) = \frac{e^{i\epsilon(p-1)}r^{p-1}}{1+r^a e^{i\epsilon}}$$

CD: $z = re^{i(2\pi-\epsilon)}$, $\rho<r<R$

$dz = dr\,e^{i(2\pi-\epsilon)}$

$$f(z) = \frac{e^{i(2\pi-\epsilon)(p-1)}r^{p-1}}{1+r^a e^{i(2\pi-\epsilon)a}}$$

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Therefore in the limit $\epsilon \to 0, \rho \to 0, R \to \infty$

$$\oint_{\Gamma} f(z)\,dz = \int_0^{\infty} \frac{r^{p-1}}{1+r^a}\,dr - \int_0^{\infty} \frac{r^{p-1}\,e^{2\pi i(p-1)}}{1+r^a\,e^{2\pi i a}}\,dr$$

$$= \int_0^{\infty} \frac{1}{2}\, r^{p-1}\left(1 - e^{2\pi i(p-1)}\right)$$

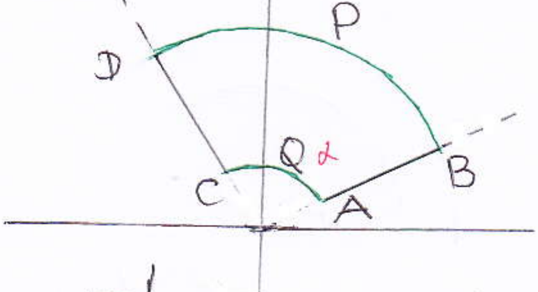
Use $AB - CD$
$= \frac{1}{2}\left[(A-C)(B+D) + (A+C)(B-D)\right]$
With $A = r^{p-1}$ $\quad B = \frac{1}{1+r^a}$
$C = r^{p-1} e^{2\pi i(p-1)}$ $\quad D = \frac{1}{1+r^a e^{2\pi i a}}$

The last integral does not get related to the required integral.

So choose radial line CD to be $\theta = 2\pi/a$ so that along CD denominator equals to the expression along $\theta = 0$

$$\left(\frac{1}{1+z^a}\right)\Bigg|_{\theta = 2\pi/a} = \frac{1}{1+r^a}.$$

Therefore we restart taking contour to be two circular arc bounded by $\theta = 0$ and $\theta = 2\pi/a$. This contour is drawn below. (Note $\alpha = 2\pi/a$)



Contour Γ' to be used for integration

AB: $\theta = \epsilon$ $\quad$ CD: $\theta = 2\pi/a$
$\rho < r < R$ $\quad$ $\rho < r < R$

Repeating the steps as before we would get

$$\oint_{\Gamma_2} \frac{z^{p-1}dz}{1+z^a} = (1-e^{2\pi i p})\int_\rho^R \frac{r^{p-1}dr}{1+r^a} = e^{i\pi p/a}(-2i\sin\pi(p/a)) \times \int_\rho^R \frac{r^{p-1}dr}{1+r^a}$$

$$\therefore \int_0^\infty \frac{x^{p-1}dx}{1+x^a} = \left(\frac{i}{2}\right)\frac{e^{-i\pi p/a}}{\sin(\pi p/a)}\oint_{\Gamma_2}\frac{z^{p-1}dz}{1+z^a}$$

Only pole at $z = e^{i\pi/a}$ is enclosed inside Γ_2 ($\because p/a < 1$)

$$\text{Res}\left(\frac{z^{p-1}}{1+z^a}\right)\Big|_{z=e^{i\pi/a}} = \lim_{z\to e^{i\pi/a}}(z-e^{i\pi/a})\frac{z^{p-1}}{(1+z^a)}$$

$$= \frac{z^{p-1}}{a\,z^{a-1}}\Big|_{z=e^{i\pi/a}} = \frac{e^{i(p-1)\pi/a}}{a\;e^{i\pi(a-1)/a}}$$

$$= \left(\frac{-1}{a}\right)e^{i\pi p/a}$$

$$\therefore \int_0^\infty \frac{x^{p-1}}{1+x^a}dx = 2\pi i \times \frac{i}{2}\frac{e^{-i\pi p/a}}{\sin(\pi p/a)} \times \left(\frac{-1}{a}\right)e^{i\pi p/a}$$

$$= \left(\frac{\pi}{a}\right)\frac{1}{\sin(\pi p/a)}$$

$$\therefore \int_0^\infty \frac{x^{p-1}}{1+x^a}dx = \frac{\pi}{a}\,\text{cosec}(\pi p/a)$$