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§§7.6 Q8 Compute $\int_0^\infty \frac{x^{1/2}}{(x^2+1)^2(x^2+4)}\,dx$ using the method of contour integration.

We take the definition

$$\sqrt{z} \overset{\text{def}}{\equiv} r^{1/2} e^{i\theta/2}, \quad 0<\theta<2\pi,$$

and integrate $f(z) \equiv \dfrac{z^{1/2}}{(x^2+1)^2(z^2+4)}$ around the closed contour Γ of fig 1. Then

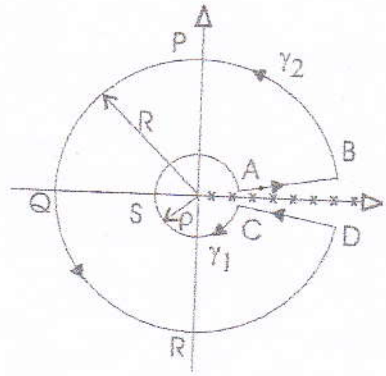

Fig. 1 Contour Γ

$$\oint_\Gamma f(z)dz = \int_{BPQRD} f(z)dz + \int_{DC} f(z)dz + \int_{CSA} f(z)dz + \int_{AB} f(z)dz \quad \text{-----}(1)$$

In the limit $\rho \to 0$, $R\to\infty$ (see fig) we have (use Darboux Thm)

$$\lim_{\rho\to 0}\int_{CSA} f(z)dz = 0 \qquad \lim_{R\to\infty}\int_{BPQRD} f(z)dz = 0.$$

Thus in this limit $\rho\to 0$, $R\to\infty$ we have from (1)

$$\lim \oint_\Gamma f(z)dz = \int_{AB} f(z)dz + \int_{DC} f(z)dz$$

$$= \int_{AB} f(z)dz - \int_{CD} f(z)dz. \quad \text{-----}(2)$$

Along the straight lines AB, CD we have

AB : $z = re^{i\epsilon}$, $\rho<r<R$

$dz = dr\,e^{i\theta}$

$$f(z) = \frac{r^{1/2}e^{i\epsilon/2}}{(z^2+1)^2(z^2+4)}$$

$\because z^{1/2} = r^{1/2}e^{i\epsilon/2}$

CD: $z = r\,e^{i(2\pi-\epsilon)}$ $\rho<r<R$

$z^{1/2} = r^{1/2}e^{(2\pi-\epsilon)i/2}$

$dz = dr\,e^{i(2\pi-\epsilon)}$ (3)

$$f(z) = \frac{r^{1/2}e^{i(2\pi-\epsilon)}}{(z^2+1)^2(z^2+4)}.$$

Therefore in the limit $\epsilon \to 0$ we will have see (2) & (3)

$$\lim \oint_\Gamma f(z)dz = \int_0^\infty \frac{r^{1/2}dr}{(r^2+1)^2(r^2+4)} - \int_0^\infty \frac{r^{1/2}e^{i\pi}dr}{(r^2+1)^2(r^2+4)}$$

$$= 2\int_0^\infty \frac{r^{1/2}dr}{(r^2+1)^2(r^2+4)}$$

$$\therefore \int_0^\infty \frac{x^{1/2}dx}{(x^2+1)^2(x^2+4)} = \frac{1}{2}\oint_\Gamma f(z)dz \qquad (4)$$

The function $f(z)$ has double poles at $z=\pm i$ and simple poles at $z=\pm 2i$. All these poles are included inside the closed contour Γ! The integral $\oint_\Gamma f(z)dz$ is computed using the residue theorem.

$$\text{Res}\{f(z)\}_{z=i} = \frac{d}{dz}\frac{(z-i)^2 z^{1/2}}{(z^2+1)^2(z^2+4)}\Bigg|_{z=i} = \frac{d}{dz}\frac{z^{1/2}}{(z+i)^2(z^2+4)}\Bigg|_{z=i}$$

$$= \frac{1}{2}\frac{z^{-1/2}}{(z+i)^2(z^2+4)} - \frac{2z^{1/2}}{(z+i)^3(z^2+4)}\Bigg|_{z=i} - \frac{z^{1/2}\times 2z}{(z+i)^2(z^2+4)^2}\Bigg|_{z=i}$$

$$= \frac{1}{2}\frac{e^{-i\pi/4}}{(-4)(3)} - \frac{2e^{i\pi/4}}{8i^3(3)} - \frac{e^{i\pi/4}\,2i}{(-4)(3)^2} \qquad \text{use } z=e^{i\pi/2} \text{ in } z^{1/2}$$

$$= \frac{-e^{-i\pi/4}}{24} - \frac{ie^{i\pi/4}}{12} + \frac{2ie^{i\pi/4}}{36}$$

$$= -\frac{e^{-i\pi/4}}{24} - \frac{ie^{i\pi/4}}{12} + \frac{ie^{i\pi/4}}{18} \qquad -\frac{1}{12}+\frac{1}{18} = \frac{-3+2}{36} = \frac{-1}{36}$$

$$= -\frac{e^{-i\pi/4}}{24} - \frac{ie^{i\pi/4}}{36} \qquad \text{---} (5)$$

3/Q8/§§7.6

$$\operatorname{Res}\{f(z)\}_{z=+2i} = \lim_{z\to +2i} (z-2i) f(z)$$

$$= \frac{z^{1/2}}{(z^2+1)^2(z+2i)}\Bigg|_{z=2i} = \frac{\sqrt{2}\, e^{i\pi/4}}{(-4+1)^2(4i)} = \frac{\sqrt{2}\, e^{i\pi/4}}{36 i} = \frac{\sqrt{2}\, e^{i\pi/4}}{36} \times(-i) \quad \text{— (6)}$$

$$\operatorname{Res}\{f(z)\}_{z=-i} = \lim_{z\to -i} \frac{d}{dz} \frac{(z+i)^2 z^{1/2}}{(z^2+1)^2(z^2+4)}\Bigg|_{z=-i}$$

$$= \frac{d}{dz} \frac{z^{1/2}}{(z-i)^2(z^2+4)}\Bigg|_{z=-i}$$ use $z = e^{i3\pi/2}$ in $z^{1/2}$

$$= \frac{1}{2} \frac{z^{-1/2}}{(z-i)^2(z^2+4)} - 2\frac{z^{1/2}}{(z-i)^3(z^2+4)} - \frac{z^{1/2}\, 2z}{(z-i)^2(z^2+4)^2}$$

$$= \frac{1}{2} \frac{e^{-3i\pi/4}}{(-4)(-1+4)} - 2\frac{e^{3i\pi/2}}{(-8i^3)(-1+4)} - \frac{e^{i3\pi/2}(-2i)}{(-4)(-1+4)^2}$$

$$= \frac{-e^{-3i\pi/4}}{24} + \frac{e^{3i\pi/4}\, i}{12} - \frac{2i\, e^{3i\pi/4}}{36}$$

$\frac{1}{12} - \frac{1}{18} = \frac{3-2}{36} = \frac{1}{36}$

$$= -\frac{e^{-3i\pi/4}}{24} + \frac{i}{36} e^{3i\pi/4} \quad \text{---- (7)}$$

$$\therefore \oint_\Gamma f(z) = 2\pi i \times \text{sum of all residues}$$

$$= 2\pi i \left[\frac{1}{24}\left(e^{-i\pi/4} - e^{-3i\pi/4}\right) + \frac{i}{36}\left(e^{i\pi/4} + e^{3i\pi/4}\right)\right]$$

Subs $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$

$$= 2\pi i \left[\frac{1}{24}\left(e^{i\pi/4} + e^{+i\pi/4}\right) + \frac{i}{36}\left(e^{i\pi/4} - e^{i\pi/4}\right)\right]$$

4/Q8/8876.

$$\mathrm{Res}\{f(z)\}_{z=-2i} = \lim_{z\to -2i} f(z)$$

$$= \frac{z^{1/2}}{(z^2+1)^2(z-2i)}\Bigg|_{z=-2i} = \frac{\sqrt{2}\, e^{+3\pi i/4}}{(-4+1)^2(-4i)}$$

use $-i \to e^{i3\pi/2}$ in $z^{1/2}$

$$= \frac{\sqrt{2}\, e^{3\pi i/4}}{-36 i} = \frac{\sqrt{2}\, i e^{3\pi i/4}}{36} = -\frac{\sqrt{2}\, e^{-\pi i/4}}{36} i \quad ---(8)$$

$\therefore$ sum of all residues in (5)–(8)

$$= \frac{(-1)}{24}\left(e^{-i\pi/4} + e^{-3i\pi/4}\right) + \frac{i}{36}\left(-e^{i\pi/4} + e^{3i\pi/4}\right)$$

$$-\frac{\sqrt{2}}{36} i\left(e^{i\pi/4} + e^{-i\pi/4}\right)$$

$e^{3i\pi/4} = -e^{-i\pi/4}$

$e^{-3\pi i/4} = -e^{i\pi/4}$

$$= \frac{-1}{24}\left(e^{-i\pi/4} - e^{i\pi/4}\right) + \frac{i}{36}\left(-e^{i\pi/4} - e^{-i\pi/4}\right)$$

$$-\frac{\sqrt{2}}{36} i\left(e^{i\pi/4} + e^{-i\pi/4}\right)$$

$$= \frac{-1}{24}(-2i)\sin(\pi/4) + \frac{i}{36}(-2\cos\pi/4) - \frac{\sqrt{2}}{36} i\, 2\cos\pi/4$$

$$= +\frac{\sqrt{2}\, i}{24} - \frac{i\sqrt{2}}{36} - \frac{i}{18}$$

$$= \frac{i}{72}\left(\sqrt{2} - 4\right)$$

$$\therefore \int_0^\infty \frac{x^{1/2}\, dx}{(x^2+1)^2(\quad +4)} = \frac{1}{2} \times 2\pi i \times \frac{i}{72}(\sqrt{2}-4)$$

$$= \frac{\pi}{72}(4-\sqrt{2})$$