6.8.2017

P1/Q7/§§7.6

$\boxed{\begin{matrix}\S\S 7.6 \\ Q7\end{matrix}}$ Compute the integral $\int_0^\infty \frac{x^p}{x^6+1}dx$, $-1<p<5$, using the method of contour integration.

We use the definition

$$f(z) = \frac{z^p}{z^6+1} = \frac{r^p e^{ip\theta}}{z^6+1}, \quad 0<\theta<2\pi$$

and set up the integration of $f(z)$ around the branch cut $\theta=0$, with contour Γ

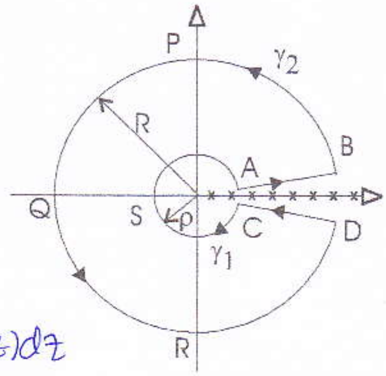

Fig. 1 Contour Γ.

$$\oint_\Gamma \frac{z^p}{z^6+1}dz = \oint_{AB} f(z)dz + \int_{BPQRD} f(z)dz + \int_{DC} f(z)dz$$

$$+ \int_{CSA} f(z)dz \quad \text{------} (1)$$

In the limit the radius $\rho\to 0$ and $R\to\infty$ the second and last integrals in (1) tend to zero. Therefore

$$\oint_\Gamma \frac{z^p}{z^6+1}dz = \int_{AB} f(z)dz + \int_{DC} f(z)dz$$

$$= \int_{AB} f(z)dz - \int_{CD} f(z)dz \quad \text{----} (2)$$

Along the straight lines AB and CD we have

AB: $z = re^{i\epsilon}, \rho<r<R$, $\quad$ CD: $z = re^{i(2\pi-\epsilon)}$ $\quad \rho<r<R$

$dz = dr\, e^{i\epsilon}$, $\quad$ $dz = dr\, e^{i(2\pi-\epsilon)}$

$f(z) = \left(\frac{r^p e^{i\epsilon p}}{r^6+1}\right)$ $\quad$ $f(z) = \frac{r^p e^{i(2\pi-\epsilon)p}}{r^6+1}$.

and $\rho<r<R$. Therefore in the limit $\epsilon\to 0$ we have from (2).

p2/Q7/§§7.6

$$\oint_\Gamma f(z)\,dz = \int_\rho^R \frac{r^p\,dr}{r^6+1} - \int_\rho^R \frac{e^{2\pi i p}\, r^p\,dr}{r^6+1} \quad \text{(in limit } \epsilon \to 0)$$

$$= (1-e^{2\pi i p}) \int_\rho^R \frac{r^p\,dr}{r^6+1}$$

Therefore

$$\int_0^\infty \frac{x^p\,dx}{x^6+1} = \lim_{\substack{\rho\to 0\\ R\to\infty}} \frac{e^{+i\pi p}}{(-2i\sin\pi p)} \oint_\Gamma \frac{z^p\,dz}{z^6+1} \quad \text{--- (3)}$$

The contour integral in the R.H.S. of (3) can be evaluated using residue theorem. The integrand has poles at roots of $z^6+1=0$. Denoting these poles by ξ_k,

$$\xi_k = e^{i\pi/6}\, e^{2\pi i k/6} \qquad k=0,1,\dots 5.$$

All the poles contibute to the integral (3). Compute residue of $f(z)$ at $z=\xi_k$

$$\text{Res}\{f(z)\}\Big|_{z=\xi_k} = \lim \frac{z^p}{(z^6+1)}(z-\xi_k)\Big|_{z=\xi_k}$$

$$= \left(\frac{pz^{p-1}\cancel{(z-\xi_k)}}{6z^5} + \frac{z^p}{6z^5}\right)\Bigg|_{z\to\xi_k}$$

Compute this limit using l' Hopital's rule

$$= \xi_k^p / 6\,\xi_k^5$$

$$= \frac{1}{6}\xi_k^{p-5}$$

While substituting ξ_k, care must be exercised to write its argument in the range 0 to 2π, as applicable to the definition of $f(z)$.

$$= \frac{1}{6} e^{i(p-5)\frac{\pi}{6}}\, e^{i\frac{2\pi k(p-5)}{6}}$$

$$k=0,1,\dots 5$$

$$= \frac{1}{6} e^{i(p-5)\frac{\pi}{6}}\, t^k \quad \text{where } t = \exp(2\pi i(p-5)/6)$$

$\therefore$ Sum of all residues

$$= \frac{1}{6} \exp\left(i(p-5)\frac{\pi}{6}\right)\left(\frac{1-t^6}{1-t}\right) \qquad t = \exp(i2\pi(p-5)/6)$$

$$= \frac{1}{6} \exp(ip\pi/6)\,(-\exp(i\pi/6)) \qquad = \exp(2\pi i(p+1)/6)$$

$$= -\frac{1}{6} e^{\frac{i(p+1)\pi}{6}} \frac{(1-e^{2\pi i p})}{1-e^{2\pi i(p+1)/6}} \qquad t^6 = \exp(2\pi i(p+1)) = \exp(2\pi i p)$$

$$= -\frac{1}{6} \frac{e^{+i\pi p}(-2i \sin \pi p/6)}{-2i \sin(p+1)\pi/6}$$

$$\therefore \oint_\Gamma \frac{z^p}{z^6+1} dz = 2\pi i \times \left(\frac{-1}{6}\right) \times \frac{e^{i\pi p} \sin(\pi p/6)}{\sin(p+1)\pi/6} \quad \text{---} ④$$

Substituting ④ in ③ we get

$$\int_0^\infty \frac{x^p\,dx}{x^6+1} = \frac{\cancel{e^{-i\pi p}}}{(-2i\,\cancel{\sin\pi p})} \times 2\pi i \times \left(-\frac{1}{6}\right) \frac{\cancel{e^{i\pi p}}\,\cancel{\sin\pi p/6}}{\sin(p+1)\pi/6}$$

$$= \frac{\pi}{6} \operatorname{cosec}(p+1)\frac{\pi}{6}.$$