5-8-2017

P1/Q5/§§7.6

§§7.6 Q5 compute the integral $\int_0^\infty \frac{x^p\,dx}{(x^2+a^2)^2}, -1<p<3$ using the method of contour integration.

Let Γ be the two circles contour of Fig 1. and $f(z)=\frac{z^p}{(z^2+a^2)^2}$, where z^p is taken to have branch cut along positive real axis. We define

$$z^p = r^p e^{ip\theta} \quad 0<\theta<2\pi.$$

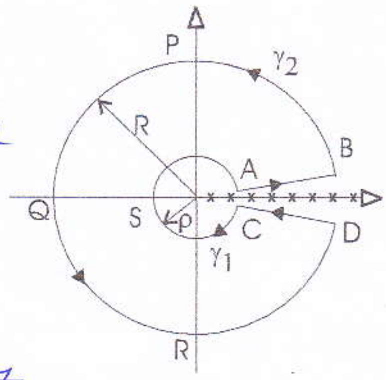

Fig. 1

Consider

$$\oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BPQRD} f(z)\,dz + \int_{DC} f(z)\,dz + \int_{CSA} f(z)\,dz$$

The circular arcs BPQRD and CSA have radii R and ρ respectively. In the limit $\rho\to 0, R\to\infty$ we have

$$\lim_{R\to\infty}\int_{BPQRD} f(z)\,dz = 0 \quad \text{and} \quad \lim_{\rho\to 0}\int_{CSA} f(z)\,dz = 0.$$

Therefore $$\oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz + \int_{DC} f(z)\,dz$$

$$= \int_{AB} f(z)\,dz - \int_{CD} f(z)\,dz. \quad \text{---- (1)}$$

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Along the straight lines AB and CD we have

AB: $z = r e^{i\epsilon}$ CD: $r e^{i(2\pi-\epsilon)}$

$dz = e^{i\epsilon} dr$ $dz = e^{i(2\pi-\epsilon)} dr$

$f(z) = \frac{e^{i\epsilon p} r^p}{(r^2+a^2)^2}$ $f(z) = \frac{e^{i(2\pi-\epsilon)p} r^p}{(r^2+a^2)^2}$

and $\rho < r < R$. Therefore in the limit $\epsilon \to 0$, we get from (1)

$$\oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz - \int_{CD} f(z)\,dz$$

$$= \int_\rho^R \frac{r^p\,dr}{(r^2+a^2)^2} - \int_\rho^R \frac{e^{i2\pi p} r^p\,dr}{(r^2+a^2)^2} \quad \text{(in limit } \epsilon \to 0)$$

$$= (1-e^{2\pi p i}) \int_\rho^R \frac{r^p\,dr}{(r^2+a^2)^2}$$

Hence

$$\int_0^\infty \frac{x^p\,dx}{(x^2+a^2)^2} = \lim_{\substack{\rho \to 0 \\ R \to \infty}} \frac{e^{-i\pi p}}{(-2i \sin \pi p)} \oint_\Gamma \frac{z^p}{(z^2+a^2)^2}\,dz \quad \text{--- (3)}$$

The contour integral in the right hand side of (3) can now be computed using the residue theorem. The integrand has two double poles at $z = \pm ia$.

$$\mathrm{Res}\left\{\frac{z^p}{(z^2+a^2)^2}\right\}_{z=ia} = \frac{d}{dz}\left\{(z-ia)^2 \frac{z^p}{(z^2+a^2)^2}\right\}_{z=ia}$$

$$= \left\{\frac{d}{dz} \frac{z^p}{(z+ia)^2}\right\}_{z=ia}$$

$$= \left(\frac{p\, z^{p-1}}{(z+ia)^2} - \frac{2 z^p}{(z+ia)^3} \right)\Bigg|_{z=ia}$$

$$\boxed{\text{use } ia = a e^{i\pi/2}}$$

$$= \left(\frac{p\, a^{p-1}\, e^{i(p-1)\pi/2}}{-4a^2} - \frac{2 a^p e^{ip\pi/2}}{8ia^3} \right)$$

The residue at $z=-ia=ae^{i3\pi/2}$ is given by

$$\operatorname{Res}\left\{ \frac{z^p}{(z^2+a^2)^2} \right\}_{z=-ia}$$

$$= \frac{d}{dz} \frac{z^p}{(z-ia)^2}\Bigg|_{z=-ia}$$

$$= \left(\frac{p z^{p-1}}{(z-ia)^2} - \frac{2z^p}{(z-ia)^3} \right)\Bigg|_{z=-ia}$$

$$= \frac{p e^{i(p-1)\frac{3\pi}{2}}}{-4a^2} - \frac{2 e^{ip\frac{3\pi}{2}} a^p}{8i a^3}$$

$\therefore J_\Gamma = 2\pi i$ sum of residues at $z=\pm ia$

$$= \frac{2\pi i\, p\, a^{p-1}}{(-4a^2)} \left(e^{i(p-1)\frac{\pi}{2}} + e^{i(p-1)\frac{3\pi}{2}} \right)$$

$$+ \frac{2\pi i\, 2a^p}{8ia^3} \left(e^{ip\pi/2} - e^{ip\frac{3\pi}{2}} \right)$$

$$= -\frac{\pi i}{2} p\, a^{p-3} e^{i(p-1)\pi} \left(e^{-i(p-1)\pi/2} + e^{+i(p-1)\pi/2} \right)$$

$$+ \frac{\pi a^{p-3}}{2} e^{ip\pi} \left(e^{-ip\pi/2} - e^{ip\pi/2} \right)$$

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$$= -\frac{\pi i p}{\cancel{2}} a^{p-3}(-e^{ip\pi})(\cancel{2}\cos(p-1)\pi/2)$$

$$+ \frac{\pi a^{p-3}}{\cancel{2}} e^{ip\pi}(-\cancel{2} i \sin p\pi/2)$$

$$= \pi i p\, a^{p-3} e^{ip\pi} \sin p\pi/2 - \pi i\, a^{p-3} e^{ip\pi} \sin \frac{p\pi}{2}$$

$$= \pi i(p-1)\, a^{p-3} e^{ip\pi} \sin(p\pi/2)$$

Using (3) we get

$$\int_0^\infty \frac{x^p\, dx}{(x^2+a^2)^2} = \left(\frac{\pi}{2}\right)(1-p)\frac{\sin(\pi p/2)}{\sin(p\pi)}\, a^{p-3}$$

$$= a^{p-3}\left(\frac{\pi}{4}\right)\sec(p\pi/2) \times (1-p)$$