p1/Q4/§§7.6

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§§7.6 Q4 Show that $\int_0^\infty \frac{x^p}{(x^2+a^2)}dx = \frac{1}{2}a^{p-1}\pi \sec\left(\frac{p\pi}{2}\right)$.

Let Γ be the contour shown in fig 1.
and $f(z) = \frac{z^p}{z^2+a^2}$, where z^p has branch cut along the positive real axis and we take

$$z^p = r^p e^{ip\theta} \qquad 0 < \theta < 2\pi$$

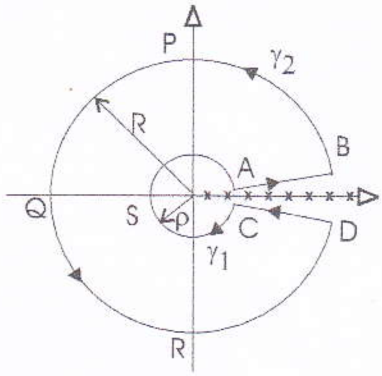

Fig. 1

Consider

$$\oint_\Gamma f(z)dz = \int_{BPQRD} f(z)dz + \int_{DC} f(z)\,dz + \int_{CSA} f(z)dz + \int_{AB} f(z)dz$$

The circular arcs BPQRD and CSA have radii R and ρ respectively. In the limit $R\to\infty$, $\rho\to 0$

$$\lim_{R\to\infty}\int_{BPQRD} f(z)dz = 0 \quad \text{and} \lim_{\rho\to 0}\int_{CSA} f(z)dz = 0$$

Therefore

$$\oint_\Gamma f(z)\,dz = \int_{AB} f(z)dz + \int_{DC} f(z)dz$$

$$= \int_{AB}\frac{z^p}{(z^2+1)}dz - \int_{CD}\frac{z^p}{(z^2+1)}dz \quad \ldots \quad (1)$$

The straight lines AB and CD have parametric equations with $\rho < r < R$.

Along the straight lines AB and CD we have

AB: $z = r e^{i\epsilon}$

$dz = e^{i\epsilon} dr$

$f(z) = \dfrac{e^{i\epsilon p} r^p}{(r^2+a^2)}$

CD: $z = r e^{i(2\pi-\epsilon)}$

$dz = e^{i(2\pi-\epsilon)} dr$

$f(z) = \dfrac{e^{i(2\pi-\epsilon)p} r^p}{(r^2+a^2)}$

and $\rho < r < R$. Therefore from (1)

$$\oint_\Gamma f(z)dz = \int_{AB} f(z)dz - \int_{CD} f(z)dz$$

$$= \int_\rho^R \frac{e^{i\epsilon p} r^p}{(r^2+a^2)} e^{i\epsilon} dr - \int_\rho^R \frac{e^{i(2\pi-\epsilon)p} r^p}{(r^2+a^2)} e^{i(2\pi-\epsilon)} dr$$

$$= \int_\rho^R \frac{r^p}{(r^2+a^2)} dr - \int_\rho^R \frac{e^{2\pi i p} r^p}{(r^2+a^2)} dr \quad \left(\text{in limit } \epsilon \to 0\right)$$

$$= (1-e^{2\pi i p}) \int_\rho^R \frac{r^p}{(r^2+a^2)} dr.$$

$$= e^{i\pi p}(-2i \sin \pi p) \times \int_\rho^R \frac{r^p}{(r^2+a^2)} dr \quad \cdots\cdots (2)$$

Hence

$$\int_0^\infty \frac{x^p}{(x^2+a^2)} dx = \lim_{\substack{\rho \to 0 \\ R \to \infty}} \frac{e^{-i\pi p}}{(-2i \sin \pi p)} \oint_\Gamma \frac{z^p}{(z^2+a^2)} dz \quad \text{---} (3)$$

→ p3

The contour integral in the right hand side of (3) can now be computed using the residue theorem. There are two poles enclosed by Γ at $z = z_1, z_2$ where

$$z_1 = ia = e^{i\pi/2} a^p \quad z_2 = -ia = e^{(2\pi - i\pi/2)i} a^p$$

(Note range of θ is 0 to 2π)

$$\mathrm{Res}\left\{\frac{z^p}{z^2+a^2}\right\}\bigg|_{z=ia} = \frac{e^{ip\pi/2}}{2ia} a^p$$

$$\mathrm{Res}\left\{\frac{z^p}{z^2+a^2}\right\}_{z=-ia} = \lim_{z\to -ia} \frac{e^{(2\pi-\pi/2)ip}}{(z^2+a^2)} (z+ia)\, a^p$$

$$= \frac{e^{i3p\pi/2}}{(-2ia)} a^{p-1}$$

$$\therefore \oint_\Gamma \frac{z^p}{(z^2+1)} dz = 2\pi i \times \left(\frac{1}{2i}\right)\left(e^{ip\pi/2} - e^{+i3\pi/2\, p}\right) a^{p-1}$$

$$= \pi e^{+i\pi p}\left(e^{-ip\pi/2} - e^{i\pi p/2}\right) a^{p-1}$$

$$= \pi e^{i\pi p}(-2i)\sin(p\pi/2)\, a^{p-1} \quad (4)$$

Substituting (4) in (3) gives

$$\int_0^\infty \frac{x^p}{(x^2+1)} dx = \frac{\pi a^{p-1} \sin(p\pi/2)}{\sin(p\pi)}$$

$$= \frac{\pi}{2} a^{p-1} \sec(p\pi/2).$$