4.8.2017

$\boxed{\begin{array}{c}\S\S 7.6\\ Q3\end{array}}$ Compute the integral $\int_0^\infty \frac{x^{p-1}}{(x+1)^2}dx$ using contour integration $(0<p<2)$.

Let z^{p-1} be defined as

$$z^{p-1} = r^{p-1} e^{i\theta(p-1)} \qquad 0<\theta<2\pi$$

So that z^{p-1} has a branch cut along the positive real axis $\theta=0$.

Next consider integral of $f(z)$ around Γ of fig. 1

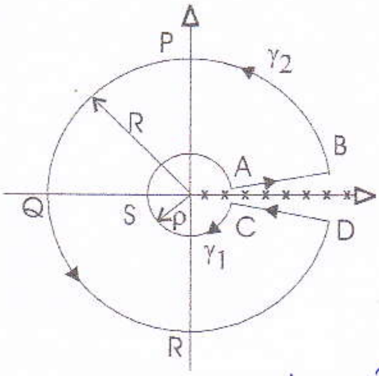

Fig. 1 Contour Γ.

$$\oint_\Gamma f(z)dz \equiv \oint_\Gamma \frac{z^{p-1}}{(z+1)^2}dz,$$

$$= \int_{AB} f(z)dz + \int_{BPQRD} f(z)dz + \int_{DC} f(z)dz + \int_{CSA} f(z)dz$$

In the limit $\rho\to 0$, $R\to\infty$ we have

$$\lim_{\rho\to 0}\int_{CSA} f(z)dz = 0 \qquad \lim_{R\to\infty}\int_{BPQRD} f(z)dz = 0$$

Therefore in this limit

$$\lim \oint_\Gamma f(z)dz = \int_{AB} f(z)dz - \int_{CD} f(z)dz$$

Along $AB: z = re^{i\epsilon}$, $\rho<r<R$

$CD: z = re^{i(2\pi-\epsilon)}$, $\rho<r<R$.

Hence $\lim \oint_\Gamma f(z)dz = \int_\rho^R \frac{r^{p-1}}{(r+1)^2}\left(e^{i(p-1)\epsilon} - e^{i(2\pi-\epsilon)(p-1)}\right)dr$

p2/Q2/SS7.6

Hence taking limit $\epsilon \to 0$,

$$\lim \oint_{\Gamma} f(z)\,dz = (1-e^{2\pi i(p-1)}) \int_0^{\infty} \frac{r^{p-1}}{(r+1)^2}\,dr$$

$$= (1-e^{2\pi i p}) \int_0^{\infty} \frac{x^{p-1}dx}{(x+1)^2}$$

$$= e^{i\pi p}(-2i)\sin\pi p \int_0^{\infty} \frac{x^{p-1}dz}{(x+1)^2}$$

or

$$\int_0^{\infty} \frac{x^{p-1}dx}{(x+1)^2} = \left(\frac{i}{2}\right) e^{-i\pi p} \operatorname{cosec}\pi p \oint_{\Gamma} f(z)\,dz \quad \text{— (1)}$$

The integral in the right hand side is evaluated by applying the residue theorem. $f(z)$ has a double pole at $z=-1$

$$\therefore \operatorname{Res}\left\{\frac{z^{p-1}}{(z+1)^2}\right\}_{z=-1} = \lim_{z\to -1} \frac{d}{dz} \frac{z^{p-1}}{(z+1)^2} (z+1)^2$$

$$= (p-1) z^{p-2}\Big|_{z=-1}$$

$$= (p-1)\, e^{i\pi(p-2)} = (p-1)e^{i\pi p}. \quad \text{— (2)}$$

$$\oint_{\Gamma} \frac{z^{p-1}}{(z+1)^2}\,dz = 2\pi i(p-1)e^{ip\pi}$$

and using (1) and (3)

$$\int_0^{\infty} \frac{x^{p-1}}{(x+1)^2}\,dx = \left(\frac{i}{2}\right) e^{-i\pi p} \operatorname{cosec}(\pi p) \times (2\pi i)(p-1)e^{i\pi p}$$

$$= (1-p)\pi \operatorname{cosec}(p\pi).$$