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§§7.6 Q2 Evaluate the integral $\int_0^\infty \frac{x^{p-1}\,dx}{x^2+2x+2}$ by the method of contour integration $(0<p<2)$

This integral can be evaluated using two circles contour Γ of Fig 1. Consider

$$\int_\Gamma \frac{z^{p-1}dz}{z^2+2z+2} = \int_{ABPQRD} + \int_{DC} + \int_{CSA} + \int_{AB}$$
$$\equiv J_\Gamma$$

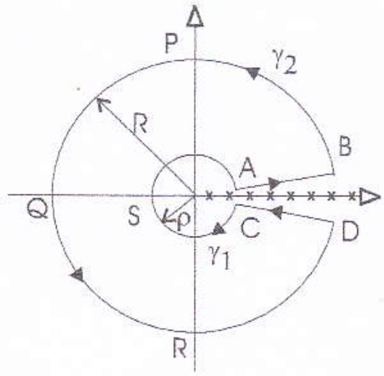

Fig. 1

In the limit $\rho \to 0$, $R\to\infty$, we will have

$$\lim_{\rho\to 0}\int_{CSA} \frac{z^{p-1}dz}{z^2+2z+2} = 0 \text{ and } \lim_{R\to\infty}\int_{BPQRD} \frac{z^{p-1}dz}{z^2+2z+2}\,dz \to 0.$$

Therefore

$$\int_\Gamma \frac{z^{p-1}dz}{z^2+2z+2} = \int_{AB} + \int_{DC} = \int_{AB} - \int_{CD}\left(\frac{z^{p-1}dz}{z^2+2z+2}\right)$$

We take the following definition of z^{p-1}

$$z^{p-1} = r^{p-1}e^{i(p-1)\theta} \qquad 0<\theta<2\pi. \qquad (1)$$

Along AB: $z = re^{i\theta}$ $\quad \rho<r<R, \quad \theta=\epsilon$

CD: $z = re^{i\theta}$ $\quad \rho<r<R \quad \theta = 2\pi-\epsilon$

$$\therefore J_\Gamma = \int_{AB} \frac{z^{p-1}dz}{z^2+2z+2} - \int_{CD}\frac{z^{p-1}dz}{z^2+2z+2}$$

$$= \int_\rho^R \frac{r^{p-1}dr}{r^2+2r+2} - \int_\rho^R \frac{r^{p-1}e^{i(2\pi-\epsilon)(p-1)}e^{i(2\pi-\epsilon)p}\,dr}{r^2+2r+2}$$

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$$J_\Gamma = (1-e^{2i\pi p})\int_\rho^R \frac{r^{p-1}\,dr}{r^2+2r+2}$$

In the limit $\rho \to 0$, $R \to \infty$ we get

$$\int_0^\infty \frac{r^{p-1}\,dr}{r^2+2r+2} = \frac{1}{1-e^{2\pi i p}} \lim J_\Gamma$$

$$= \frac{-e^{-\pi i p}}{-2i\sin\pi p} J_\Gamma$$

$$1-e^{2\pi i p} = e^{\pi i p}(e^{-i\pi p}-e^{\pi i p}) = -2i\sin\pi p\, e^{i\pi p}.$$

$$= \left(\frac{i}{2}\right)\operatorname{cosec}(\pi p)\, e^{i\pi p} J_\Gamma \qquad (2)$$

In the limit $\rho \to 0$, $R \to \infty$, the contour Γ encloses two simple poles given by $z^2+2z+2=0$. or z

$$z = \frac{-2 \pm \sqrt{4-4\times 2\times 1}}{2} = \frac{-2\pm 2i}{2}$$

$$= (-1 \pm i)$$

Thus the two roots are

$$z_1 = (-1+i) = \sqrt{2}\, e^{i3\pi/4} \qquad z_2 = (-1-i) = \sqrt{2}\, e^{i5\pi/4} \qquad (3)$$

In order to evaluate the residues the arguments of z_1 and z_2 must be kept in range $(0, 2\pi)$ See Eq (1).
Writing $z^2+2z+2 = (z-z_1)(z-z_2)$ the required residues are easily computed.

$$\left.\begin{aligned} \operatorname{Res}\left\{\frac{z^{p-1}}{z^2+2z+2}\right\}_{z=z_1} &= \frac{2^{(p-1)/2}\, e^{i\frac{3\pi}{4}(p-1)}}{(z_1-z_2)} \\ \operatorname{Res}\left\{\frac{z^{p-1}}{z^2+2z+2}\right\}_{z=z_2} &= \frac{2^{(p-1)/2}\, e^{i\frac{5\pi}{4}(p-1)}}{(z_2-z_1)} \end{aligned}\right\} \quad (4)$$

$J_\Gamma = 2\pi i \times$ sum of residues at $z = z_1$ and $z = z_2$

$$= (2\pi i)\, 2^{(p-1)/2} \left(\frac{e^{i\frac{3\pi}{4}(p-1)} - e^{i\frac{5\pi}{4}(p-1)}}{2i} \right) \qquad z_1 - z_2 = 2i$$

$$= (\cancel{2}\pi \cancel{i})\, 2^{(p-1)/2} e^{i\pi(p-1)} \left(\frac{1}{\cancel{2i}}\right) \left(e^{-i\pi(p-1)/4} - e^{i\pi(p-1)/4} \right)$$

$$= \pi\, 2^{(p-1)/2} e^{i\pi(p-1)}\, 2i \sin\frac{(1-p)\pi}{4} \qquad (5)$$

$\therefore$ Using (2) and (5) we get the required answer

$$\int_0^\infty \frac{x^{p-1}\,dx}{x^2+2x+2} = \left(\frac{i}{\cancel{2}}\right) \operatorname{cosec}(\pi p)\, e^{i\pi(p-1)} \times \pi\, 2^{(p-1)/2} e^{i\pi(p-1)} \times \cancel{2} i \sin(1-p)\frac{\pi}{4}$$

$$= \pi\, 2^{(p-1)/2} \operatorname{cosec}(p\pi) \sin(1-p)\frac{\pi}{4}$$

$$\therefore \int_0^\infty \frac{x^{p-1}\,dx}{x^2+2x+2} = 2^{(p-1)/2}\, \pi \operatorname{cosec}(p\pi) \sin\frac{(1-p)\pi}{4}.$$