P1/Q1/§§7.6 Q1/1

1.8.2017

§§7.6 Q[1] Evaluate the integral $\int_0^\infty \frac{x^{1/2}\,dx}{(x+4)(x+25)}$ using the two circles contour shown in Fig. 1.

<u>Solution</u> Consider the integral

$$J_\Gamma = \oint_\Gamma \frac{z^{1/2}\,dz}{(z+4)(z+25)}$$

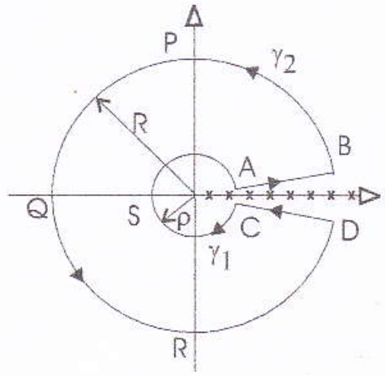

Fig. 1 contour Γ

Writing the integral as sum of four parts we get

$$J_\Gamma = \int_{AB} + \int_{BPQRD} + \int_{DC} + \int_{CSA}$$

where BPQRD and CSA are circular arcs of radii R and ρ as shown in the figure. We will take the limit $\rho \to 0$ and $R \to \infty$. In these limits we would get

$$\lim_{R\to\infty} \int_{BPQRD} \frac{z^{1/2}\,dz}{(z+4)(z+25)} \longrightarrow 0$$

and

$$\lim_{\rho\to 0} \int_{CSA} \frac{z^{1/2}\,dz}{(z+4)(z+25)} \longrightarrow 0$$

Therefore

$$J_\Gamma = \int_{AB} \frac{z^{1/2}\,dz}{(z+4)(z+25)} + \int_{DC} \frac{z^{1/2}\,dz}{(z+4)(z+25)}$$

We take the definition of $z^{1/2}$ to be

$$z^{1/2} = r^{1/2} e^{i\theta/2}, \quad 0 < \theta < 2\pi .$$

Along AB we have $z = r e^{i\epsilon}$ $\rho \le r \le R$

Therefore, in the limit $\epsilon \to 0, \rho \to 0, R \to \infty$, we have

$$\int_{AB} \frac{z^{1/2}\,dz}{(z+4)(z+25)} = \int_0^\infty \frac{r^{1/2}\,dr}{(r+4)(r+25)}$$

Also along CD $z = r e^{i(2\pi - \epsilon)}$ $\rho \le r \le R$

Hence

$$\int_{CD} \frac{z^{1/2}\,dz}{(z+4)(z+25)} = \int_0^\infty \frac{r^{1/2} e^{i\pi}\,dr}{(r+4)(r+25)}$$

holds in the limit $\epsilon \to 0, \rho \to 0, R \to \infty$.

$$\therefore \lim J_\Gamma = \lim \int_{AB} \frac{z^{1/2}\,dz}{(z+4)(z+25)} - \lim \int_{CD} \frac{z^{1/2}\,dz}{(z+4)(z+25)}$$

$$= \lim \int_0^\infty \frac{2r^{1/2}\,dr}{(r+4)(r+25)}$$

Thus the required integral is given by

$$\int_0^\infty \frac{r^{1/2}\,dr}{(r+4)(r+25)} = \frac{1}{2} \lim \int_\Gamma \frac{z^{1/2}\,dz}{(z+4)(z+25)}$$

The closed contour $\Gamma$ in the right hand side encloses poles at $z = -4, -25$. The residues at these points are easily calculated.

$$\operatorname{Res}\left\{\frac{z^{1/2}}{(z+4)(z+25)}\right\}_{z=-4} = \lim_{z \to -4} (z+4)\frac{z^{1/2}}{(z+4)(z+25)}$$

$$= \frac{4^{1/2} e^{i\pi/2}}{(21)} = \frac{2i}{21} \qquad z = -4 = 4e^{i\pi}$$

Similarly the residue at $z = -25 = 25e^{i\pi}$ is

$$\text{Res}\left\{\frac{z^{1/2}}{(z+4)(z+25)}\right\}_{z=-25} = \frac{5e^{i\pi/2}}{-21}$$

$$\therefore J_\Gamma = 2\pi i \times \left(\frac{2i}{21} - \frac{5i}{21}\right)$$

$$= 2\pi i \times \frac{-3i}{21} = 2\pi/7$$

$\therefore$ Required integral $= (1/2) J_\Gamma = \frac{\pi}{7}$