§§7.4 Q5

Compute $\int_0^\infty \frac{x^2 \cos px \, dx}{x^4+1}$ using the method of contour integration.

We first change $\cos px$ writing it as

$$\cos px = \cancel{\frac{e^{ipx} + e^{-ipx}}{2}} = \mathrm{Re}\, e^{ipx}$$

$$\therefore \int_0^\infty \frac{x^2 \cos px \, dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 \cos px}{x^4+1} dx$$

$$= \frac{1}{2}\mathrm{Re} \int_{-\infty}^{\infty} \frac{x^2 e^{ipx}}{x^4+1} dx$$

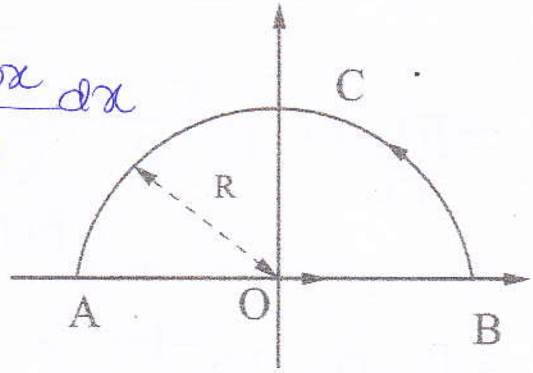

Fig. 1 Semi-circular contour

This integral is equal to a contour integral along AOBCA in the limit $R \to \infty$.

$$\int_0^\infty \frac{x^2 \cos px}{x^4+1} dx = \frac{1}{2}\mathrm{Re} \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2 e^{ipx}}{x^4+1} dx$$

$$= \frac{1}{2}\mathrm{Re} \lim_{R\to\infty} \oint_{AOBCA} \frac{z^2 e^{ipz}}{(z^4+1)} dz.$$

The integrand has four poles at $e^{i\pi/4}$, $e^{3i\pi/4}$, $e^{-i\pi/4}$, $e^{-i3\pi/4}$

Of these first two lie in the upper half plane and contribute to the integral. Computing residues at these points.

$$\mathrm{Res} \left.\frac{z \cdot e^{ipz}}{(z^4+1)}\right|_{z=\xi_{1,2}} = \left.\lim \frac{(z-\xi)\, z^2\, e^{ipz}}{(z^4+1)}\right| \quad \text{where } \xi_{1,2} = e^{i\pi/4}, e^{i3\pi/4} = \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$$

$$= \lim_{z\to\xi} z^2 e^{ipz} \lim_{z\to\xi} \frac{z-\xi}{(z^4+1)} = \xi^2 e^{ip\xi}\left(\frac{1}{4\xi^3}\right)$$

$$= \frac{1}{4} e^{ip\xi} \frac{1}{\xi}$$

$$\therefore \oint_{AOBCA} \frac{z^2 \cos pz}{(z^4+1)}\, dz$$

$$= 2\pi i\left(\frac{1}{4}\frac{1}{\xi_1} e^{ip\xi_1} + \frac{1}{4}\xi_2 e^{ip\xi_2}\right) \qquad \xi_1 = \frac{1+i}{\sqrt{2}}$$

$$= \frac{2\pi i}{4\xi_1\xi_2}\left(e^{-p/\sqrt{2}+ip/\sqrt{2}}\xi_2 + \xi_1 e^{-p/\sqrt{2}-ip/\sqrt{2}}\right) \qquad \xi_2 = -\frac{1+i}{\sqrt{2}} \qquad \xi_1\xi_2 = -1$$

$$= -\frac{\pi i}{2} e^{-p/\sqrt{2}}\left(e^{ip/\sqrt{2}}\xi_2 + e^{-ip/\sqrt{2}}\xi_1\right)$$

$$= -\frac{\pi i}{2} e^{-p/\sqrt{2}}\left(\cos(p/\sqrt{2})(\xi_2+\xi_1) + i\sin p/\sqrt{2}\,(\xi_2-\xi_1)\right)$$

$$= \frac{\pi}{2} e^{-p/\sqrt{2}}\left(-i\cos(p/\sqrt{2})\, i\sqrt{2} - \sin(p/\sqrt{2}) \times \sqrt{2}\right)$$

$$= \frac{\pi}{\sqrt{2}} e^{-p/\sqrt{2}}\left(\cos(p/\sqrt{2}) - \sin(p/\sqrt{2})\right)$$

$\therefore$ Required integral

$$= \frac{1}{2}\,\mathrm{Im} \oint_{AOBCA} \frac{z^2\cos pz}{(z^4+1)}\, dz$$

$$= \frac{\pi}{2\sqrt{2}} e^{-p/\sqrt{2}}\left(\cos(p/\sqrt{2}) - \sin(p/\sqrt{2})\right)$$