9.12.2017

Q14

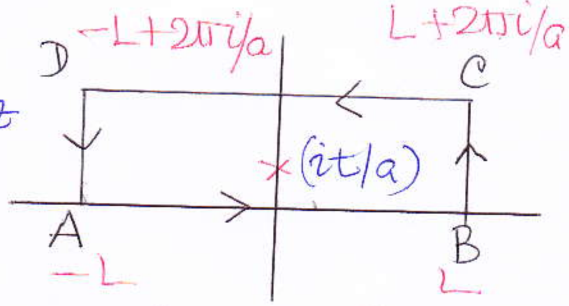
§879 Compute the integral

$$\int_0^\infty \frac{dx}{\cosh ax + \cos t}\,dx$$

using contour integration.

We will integrate $f(z) = \dfrac{z}{\cosh az + \cos t}$ around rectangular contour Γ shown in figure. The corners of the rectangular contour are taken to be at: $-L$, L, $L + \frac{2\pi i}{a}$, $-L + \frac{2\pi i}{a}$.

$$\oint_\Gamma f(z)\,dz = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} f(z)\,dz$$



Contour Γ.

The integrals along BC and DA tend to zero as $L \to \infty$. Dropping these terms.

$$\lim \oint_\Gamma f(z)\,dz = \int_{AB} f(z)\,dz + \int_{CD} f(z)\,dz$$

$$= \int_{AB} f(z)\,dz - \int_{DC} f(z)\,dz \quad \text{------} \quad (1)$$

(We take t to be between $-\pi$ and π without any loss of generality.)

The function $f(z)$ has pole at $z = t/a$ lying ~~out~~ inside the contour Γ

Since along AB $z = x$, $-L \le x \le L$ and along DC $z = x + \frac{2\pi i}{a}$, $-L \le x \le +L$, from Eq (1) we get

$$\int_{-\infty}^{\infty} \frac{x\,dx}{\cosh ax + \cos t} - \int_{-\infty}^{\infty} \frac{(x + 2\pi i/a)\,dx}{\cosh ax + \cos t} = \oint_{\Gamma} f(z)\,dz$$

or $$-\frac{2\pi i}{a} \int_{-\infty}^{\infty} \frac{dx}{\cosh ax + \cos t} = \oint_{\Gamma} f(z)\,dz$$

or $$\int_{-\infty}^{\infty} \frac{dx}{\cosh ax + \cos t} = \frac{-a}{2\pi i} \oint f(z)\,dz$$

$$= a \times \text{Residue} \{f(z)\}_{z = it/a}$$

$$= a \times \lim_{z \to it/a} z \left(\frac{z - it/a}{\cosh az + \cos t} \right)$$

$$= a \cdot \left(\frac{it}{a} \right) \times \frac{1}{a} \frac{1}{\sinh(it)}$$

$$= \frac{t}{a} \operatorname{cosec} t$$