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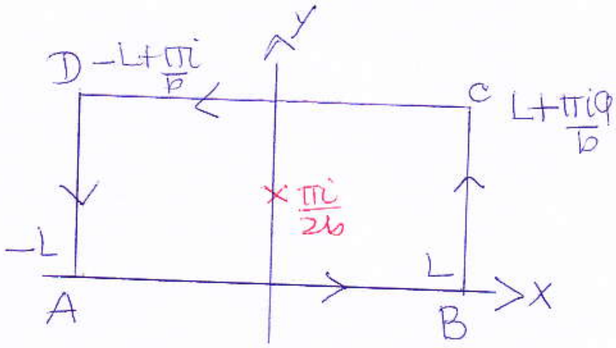
Q12
§§79 Integrate $\int_0^\infty \frac{\cosh ax}{\cosh bx}\,dx$, $0<a<b$.

We consider a rectangular contour ABCDA with corners at $(-L,0)$, $(L,0)$ $(L,\pi/b,0)$ $(-L,\pi a/b)$ and integrate $\frac{\exp(az)}{\cosh bz} \equiv f(z)$ around ABCDA.



$$\oint_{ABCDA} f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

We shall take the limit $L \to 0$. In this limit $\int_{BC}$ and $\int_{DA}$ tend to zero. Hence

$$\oint_{ABCDA} f(z)\,dz = \int_{AB} f(z)\,dz - \int_{DC} f(z)\,dz$$

$\because \cosh(\pi i+z) = -\cosh z$ (and) Along DC $z = x+\frac{\pi i}{b}$

$$= \int_{-\infty}^{\infty} \frac{e^{ax}\,dx}{\cosh bx} + \int_{-\infty}^{\infty} \frac{e^{i\pi a/b}\,e^{ax}}{\cosh bx}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh bx}\,dx \times (1+e^{i\pi a/b})$$

$$= (1+e^{i\pi a/b}) \times 2\int_0^\infty \frac{\cosh ax}{\cosh bx}\,dx \quad \text{---(1)}$$

The contour integral of $f(z)$ around ABCDA is given by

$$\oint_{ABCDA} \frac{e^{az}}{\cosh bz}\,dz = 2\pi i \times \text{Res}\left\{\frac{e^{az}}{\cosh bz}\right\}_{z=\frac{\pi i}{2b}}$$

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$$= 2\pi i \times \lim_{z \to \frac{\pi i}{2b}} e^{az} \frac{(z - \pi i/2b)}{\cosh(bz)}$$

$$= 2\pi i \times e^{i\pi a/2b} \times \frac{1}{b \times \sinh(\frac{i\pi}{2})} \qquad \because \lim \frac{z - \pi i/2b}{\cosh(zb)} = \frac{1}{b \sin \pi i/2}$$

$$= \frac{2\pi i \, e^{i\pi a/2b}}{i b} = \frac{\pi}{b} e^{i\pi a/2b} \quad \text{---(2)}$$

Substituting answer (2) in (1) we get

$$2(1+e^{i\pi a/b}) \int_0^\infty \frac{\cosh ax}{\cosh bx} dx = \frac{2\pi}{b} e^{i\pi a/2b}$$

$$\int_0^\infty \frac{\cosh ax}{\cosh bx} = \frac{\pi}{b} \frac{e^{\pi i a/2b}}{1+e^{\pi i a/b}} = \frac{\pi}{b} \times \frac{1}{(e^{-\pi i a/2b} + e^{\pi i a/2b})}$$

$$\therefore \int_0^\infty \frac{\cosh ax}{\cosh bx} dx = \frac{\pi}{2b} \sec(\pi a/2b)$$