Q7–Q10 §§7.9 Successively prove the results

$$\int_0^\infty \frac{x^2\,dx}{\cosh\pi x} = \frac{1}{8}, \quad \int_0^\infty \frac{x^4\,dx}{\cosh\pi x} = \frac{5}{32},$$

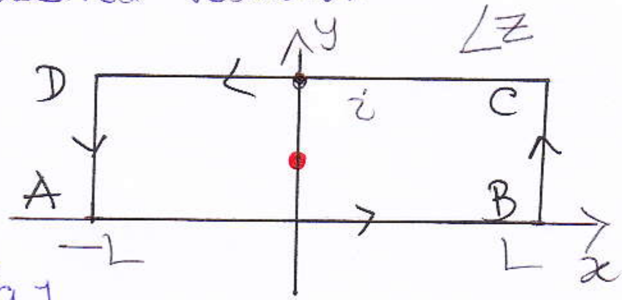
$$\int_0^\infty \frac{x^6\,dx}{\cosh\pi x} = \frac{61}{128}, \quad \int_0^\infty \frac{x^8\,dx}{\cosh\pi x} = \frac{1385}{512}.$$

These results will be proved by deriving reduction formula for

$$I_n = \int_{-\infty}^{\infty} \frac{x^n}{\cosh\pi x}\,dx \qquad \text{------ (1)}$$

Note that $I_{2n+1} = 0$. To obtain desired reduction formula consider the integral

$$J_n = \oint_\Gamma \frac{z^n}{\cosh\pi z}\,dz$$



around rectangular contour of Fig 1.

In the limit $L \to \infty$ we will have

$$\int_{DA} \frac{z^n}{\cosh\pi z}\,dz \to 0 \quad \text{and} \quad \int_{BC} \frac{z^n}{\cosh\pi z}\,dz = 0$$

$$\therefore J_n = \int_{AB} \frac{z^n\,dz}{\cosh\pi z} - \int_{DC} \frac{z^n}{\cosh\pi z}\,dz$$

[$\because$ on DC, $z = x+i$, $\cosh\pi(x+i) = -\cosh\pi x$

$$= \int_{-\infty}^{\infty} \frac{x^n}{\cosh\pi x}\,dx + \int_{-\infty}^{\infty} \frac{(x+i)^n}{\cosh\pi x}\,dx \qquad \text{---- (2)}$$

The contour integral J_n is computed using the residue theorem. The contour Γ encloses only one singular point of integrand $(z^n/\cosh\pi z)$. It is a simple pole at $z = i/2$ Hence

$$J_n = 2\pi i \times \mathrm{Res}\left\{\frac{z^n}{\cosh\pi z}\right\}_{z=i/2}$$

$$= 2\pi i \lim_{z\to i/2}\left(z^n \frac{(z-i/2)}{\cosh\pi z}\right)$$

$$= 2\pi i \lim_{z\to i/2} z^n \lim_{z\to i/2}\left(\frac{z-i/2}{\cosh\pi z}\right)$$

$\because$ limit of products is product of limit

$$= (2\pi i)(i/2)^n \frac{1}{\pi\sinh\pi i/2}$$

Use L Hopital's rule to compute the second limit.

$$= \left(\frac{i^n}{2^{n-1}}\right) \quad \text{------ (3)}$$

Using (3) in (2) gives

$$\int_{-\infty}^{\infty}\frac{(x+i)^n\,dx}{\cosh\pi x} + \int_{-\infty}^{\infty}\frac{x^n}{\cosh\pi x}\,dx = \left(\frac{i^n}{2^{n-1}}\right) \quad \text{---- (4)}$$

Noting $I_n = 0$ for odd n, we use (4) successively setting $n = 2, 4, 6, 8$.

<u>For $n=0$</u> $\quad 2\int_{-\infty}^{\infty}\frac{dx}{\cosh\pi x} = 2 \quad , \; I_0 = 1$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{\cosh\pi x} = 1 \implies \int_{0}^{\infty}\frac{dx}{\cosh\pi x} = 1/2$$

Setting $n=2$ in (4) gives

$$2I_2 + i^2 I_0 = -1/2 \Rightarrow I_2 = 1/4$$

and $\int_0^\infty \frac{x^2\,dx}{\cosh \pi x} = \frac{1}{2} I_2 = 1/8$

For $n=4$, relation (4) reduces to

$$2I_4 + \frac{4(4-1)}{2!} i^2 I_2 + I_0 = \frac{1}{8}$$

$\therefore I_4 = \frac{5}{16} \Rightarrow \int_0^\infty (x^4/\cosh \pi x)\,dx = \frac{1}{2} I_4 = \frac{5}{32}$

Next taking $n=6$, we get

$$2I_6 + i^2 \frac{6(6-1)}{2} I_4 + \frac{6(6-1)}{2} I_2 - I_0 = \frac{-1}{32}$$

$I_6 = \frac{61}{64}$ and $\int_0^\infty \frac{x^6\,dx}{\cosh \pi x} = \frac{61}{128}$

Result (4) for $n=8$ gives

$$2I_8 - \frac{8(8-1)}{2} I_6 + \frac{8(8-1)(8-2)(8-3)}{4!} I_4 - \frac{8(8-1)}{2} I_2 + I_0 = J_8$$

or $2I_8 - 28I_6 + 70I_4 \ 28I_2 + I_0 = \frac{1}{128}$

This gives $I_8 = \frac{1385}{256}$

Hence $\int_0^\infty \frac{x^8}{\cosh \pi x}\,dx = \frac{1385}{512}$