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§§7.9 Q6

p1/Q6/§§7.9

Compute the integral

$$\int_0^\infty x\left(\frac{\exp(-ax)+\exp((a-p)x)}{1-\exp(-px)}\right)dx$$

using the method of contour integration.

In order to use a hyperbolic contour we need to extend the range of integration from 0 to ∞ to entire real axis $-\infty < x < \infty$.

For this purpose we note that

$$\int_0^\infty x\,\frac{\exp(-ax)}{1-\exp(-px)}\,dx = \int_{-\infty}^{0}(-x)\,\frac{\exp(ax)}{1-\exp(px)}\,dx$$

$$= \int_{-\infty}^{0} x\,\frac{\exp((a-p)x)}{1-\exp(-px)}\,dx$$

Therefore given integral becomes

$$\int_0^\infty x\,\frac{\exp(-ax)+\exp((a-p)x)}{1-\exp(-px)}\,dx$$

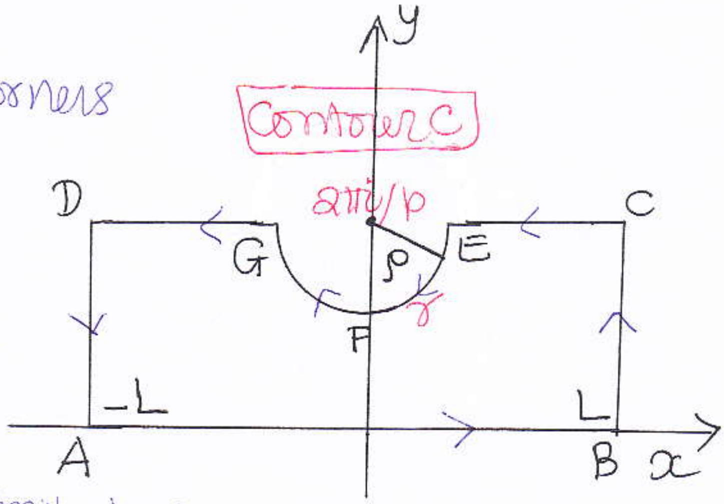
$$= \int_{-\infty}^{\infty} x\,\frac{\exp((a-p)x)}{1-\exp(-px)}\,dx = \int_{-\infty}^{\infty} x\,\frac{\exp(ax)}{\exp(px)-1}\,dx$$

To compute the integral

$$I = \int_{-\infty}^{\infty} x\,\frac{\exp(ax)}{\exp(px)-1}\,dx$$

We set up contour integral of $f(z) = z\,\dfrac{\exp(az)}{\exp(pz)-1}$ around the contour C shown in figure on the next page.

In the figure A, B, C, D are corners of a rectangle at $(-L,0)$, $(L,0)$, $(L, \frac{2\pi i}{p})$ and $(-L, \frac{2\pi i}{p})$. EFG is a semicircle with center at $2\pi i$ and radius ρ.



We shall be interested in limit $L \to \infty$ and $\rho \to 0$. In this limit

$$\int_{DA} f(z)dz = \int_{BC} f(z)dz = 0$$

$$\int_{EFG} f(z)dz = \underline{-\pi i} \operatorname{Res}\{f(z)\}\Big|_{z=2\pi i/p}$$

Note factor $-\pi i$ appears here. See §6.5.1

The function $f(z)$ is analytic inside and on the closed contour C. Therefore $\oint_C f(z)dz = 0$. In the limit $\rho \to 0$, $L \to \infty$ we get contributions from horizontal line segments and semicircular arc. Thus we would get

$$\lim \oint f(z)dz = \int_{-\infty}^{\infty} f(x)\,dx - \int_{-\infty}^{\infty} f(x + \tfrac{2\pi i}{p})\,dx + \int_{EFG} f(z)dz = 0$$

This is because along AB $z = x$, $dz = dx$ $-L < x < L$ and along DGUEC $z = x + \frac{2\pi i}{p}$ where $x \in (-L, L)$. in limit $\rho \to 0$

p3/Q6/8879

Thus we get

$$\int_{-\infty}^{\infty} x \frac{e^{ax}}{e^{px}-1}\,dx - e^{2\pi i a/p}\int_{-\infty}^{\infty}\frac{(x+2\pi i/p)\,e^{ax}}{e^{px}-1}\,dx$$

$$+\int_{\gamma}\frac{z\,e^{az}}{e^{pz}-1}\,dz = 0$$

$$I\,(1-\exp(2\pi i a/p)) - \left(\frac{2\pi i}{p}\right)\int_{-\infty}^{\infty}\frac{e^{ax}}{e^{px}-1}\,dx \times e^{2\pi i a p}$$

$$-\pi i\,\mathrm{Res}\left(\frac{z\,e^{az}}{e^{pz}-1}\right)\Big|_{z=\frac{2\pi i}{p}} = 0 \qquad ----(1)$$

Computing the required residue

$$\mathrm{Res}\left(\frac{z\,e^{az}}{e^{pz}-1}\right)\Big|_{z=2\pi i/p} = \frac{2\pi i}{p}\,\frac{e^{2\pi i a/p}}{p}$$

Therefore Eq (1) becomes

$$I\,(1-\exp(2\pi i a/p)) - \frac{2\pi i}{p}\exp(2\pi i a/p)\int_{-\infty}^{\infty}\frac{e^{ax}}{e^{px}-1}\,dx$$

$$+\frac{2\pi^2}{p^2}\exp(2\pi i a/p)$$

or $$I\,(e^{(-2\pi i a/p)}-1) - \frac{2\pi i}{p}\int_{-\infty}^{\infty}\frac{e^{ax}}{e^{px}-1}\,dx$$

$$+\frac{2\pi^2}{p^2} = 0.$$

Equating real part of the above eqn to zero gives

$$I\left(\cos\left(\frac{2\pi a}{p}\right)-1\right)+\frac{2\pi^2}{p^2}=0$$

or $$I = \frac{\pi^2}{p^2}\,\mathrm{cosec}^2\left(\frac{\pi a}{p}\right).$$