30-11-2017 P1/Q5/§§7.9

Q5 §§7.9 Compute the following integral using the method of contour integration.

$$\int_0^\infty \frac{x(1-e^{-x})e^{-x}}{(1+e^{-3x})}dx$$

First we shall convert the integral into an integral over entire range $(-\infty,\infty)$. Using I to denote the required integral

$$I=\int_0^\infty \frac{x(1-e^{-x})e^{-x}}{(1+e^{-3x})}dx = \int_{-\infty}^{0} \frac{(-x)(1-e^{x})e^{x}dx}{(1+e^{3x})}$$

$$=\int_{-\infty}^{0}\frac{x(e^{x}-1)e^{x}dx}{(e^{3x}+1)} = \int_{-\infty}^{0}\frac{x(1-e^{-x})e^{-x}}{(1+e^{-3x})}dx$$

$$\therefore I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x(1-e^{-x})e^{-x}}{(1+e^{-3x})}dx.$$

To compute this integral we set up

$$J=\oint_{ABCDA} f(z)dz$$

along rectangular contour ABCDA, with CD being line $\operatorname{Im} z = 2\pi$ and taking

$$f(z)=\frac{z^2(1-e^{-z})e^{-z}}{(1+e^{-3z})}$$

Reader should verify why choice $\frac{z(1-e^{-z})e^{-z}}{1+e^{-3z}}$ does not work.

P2/Q5/887.9

The integral J can be computed using residue theorem.

The contour encloses poles at $z = i\pi/3,\ i\pi,\ \pi i/3$.

Computing the residues

$z_1 = i\pi/3$

$z_2 = i\pi$

$z_3 = 5\pi i/3$

$$R_k = \operatorname{Res}\left\{ \frac{z^2(1-e^{-z})e^{-z}}{(1+e^{-3z})} \right\}_{z=}$$

$$= z_k^2(1-e^{-z_k})e^{-z_k} \times \left(\frac{z-z_k}{1+e^{-3z}}\right)\Bigg|_{z\to}$$

$$= z_k^2(e^{-z_k}-e^{-2z_k}) \times \lim_{z\to z_k}\left(\frac{z-z_k}{1+e^{-3z}}\right)$$

$$\hookrightarrow \frac{1}{-3e^{-3z}}\Bigg|_{z=z_k}$$

$$= z_k^2(e^{-z_k}-e^{-2z_k}) \times \left(\frac{1}{+3}\right)$$

$$R_1 = \left(\frac{-\pi^2}{9}\right) \times \frac{1}{3} \times \left(e^{-i\pi/3} - e^{-2\pi i/3}\right)$$

$$= -\frac{\pi^2}{9} \times \frac{1}{3} \times 1 = \frac{-\pi^2}{27}$$

$$R_2 = (-\pi^2) \times \frac{1}{3} \times (e^{-i\pi} - e^{-2i\pi}) = \frac{2\pi^2}{3}$$

$$R_3 = \left(-\frac{25\pi^2}{9}\right) \times \frac{1}{3} \times \left(e^{-5\pi i/3} - e^{-10\pi i/3}\right)$$

$$= \frac{-25\pi^2}{27}$$

$-5\pi i/3$
$= 2\pi i - \pi i/3$

$-10\pi i/3 = -\pi i - \pi i/3$

$$\therefore J = 2\pi i \times (R_1 + R_2 + R_3)$$

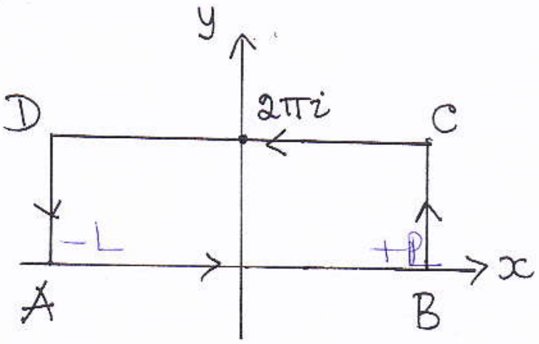
$$= 2\pi i\,(\pi^2/27)(-1 + 18 - 25)$$

$$= (2\pi i)\left(-\frac{8\pi^2}{27}\right) \quad \text{------} \quad (1)$$

P3/Q5/§87.9

The contour ABCDA is a closed rectangular contour with vertices at $(-L,0)$, $(0,L)$, $(L,2\pi i)$, $(L,-2\pi)$.

The integrals along DA and BC can be shown to vanish as $L \to \infty$.



Hence

$$J = \oint_{ABCDA} f(z)\,dz = \int_{AB} f(z)\,dz - \int_{DC} f(z)\,dz$$

In the limit $L \to \infty$

$$J = \int_{-\infty}^{\infty} f(x)\,dx - \int_{-\infty}^{\infty} f(x+2\pi i)\,dx$$

$$= -4\pi i \int_{-\infty}^{\infty} \frac{x(1-e^{-x})e^{-x}}{(1+e^{3x})}\,dx + 4\pi^2 \int_{-\infty}^{\infty} \frac{(1-e^{-x})e^{-x}}{(1+e^{3x})}\,dx$$

$$= -8\pi i \int_{0}^{\infty} \frac{x(1-e^{-x})e^{-x}}{(1+e^{3x})}\,dx + 4\pi^2 \int_{-\infty}^{\infty} \frac{(1-e^{-x})e^{-x}}{(1+e^{3x})}\,dx \quad (2)$$

comparing real and imaginary parts of (1) and (2) we get

$$\int_0^{\infty} \frac{x(1-e^{-x})e^{-x}\,dx}{(1+e^{3x})} = \frac{2\pi^2}{27}$$